SCSVMV

Department of Mathematics

Course material

II ECE

CALCULUS & SPECIAL FUNCTIONS

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Sub.Code : CBSMAJ8T10	**Mathematics – IV** Calculus, Special Functions and Statistics	L T P - 3 1 0	Credits-04

Mathematics – IV

(B.E./B.Tech. Engineering FOURTH SEMESTER (2018 - 2022 BATCH)

This course focuses on the topics in Calculus, Ordinary Differential Equations of higher order and Designs of Experiment. The fundamentals and the way to solve Ordinary differential equation problems are introduced. Understanding the basic concepts and their properties are important for the development of the present and further courses.

Unit I: Calculus

Homogeneous Functions-Total derivative-Change of variables-Jacobian-Taylor's theorem for function of two variables-Maxima and Minima of functions of two variables-Lagranges method of undermined multipliers

Unit II: Multi Variable Calculus

Directional derivatives-Gradient-curl and divergence-Problems on Green-Gauss and Stokes theorems- orthogonal curvilinear coordinates-Simple applications involving cubes, sphere and rectangular parallelepipeds.

Unit III: Special Functions –I

Validity of series solution - Series solution when x=0 is an ordinary point - Frobenius method (Series solution when x=0 is a regular singularity) - Bessel's equation (Bessel's functions of the first and second kind) - Recurrence formulae for Jn(x) - Expansions for J0 and J1: Value of J1/2 - Generating function for Jn(x) - Equations reducible to Bessel's equation – Orthogonality of Bessel functions

Unit IV: Special Function-II

Legendre's Equation – Rodrigue's Formula – Legendre Polynomials – Generating Function for $P_n(x)$- Recurrence formula for $P_n(x)$-Orthogonality of Legendre Polynomials – Hermite Polynomials-Recurrence formulae-Rodrigue's formula-Orthogonality of Hermite polynomials

Unit V: Design of Experiment

Design of experiments – Completely randomized design: Analysis of variance for one factor of classification – Randomized block design: Analysis of variance for two factors of classification – Latin square design.

Suggested Books:

1. Grewal B.S, Higher Engineering Mathematics, 41st Edition, Khanna Publishers, New Delhi, 2011.

2. Gupta S.P, Statistical Methods, 28th Edition, Sultan Chand and Sons., New Delhi, 1997.
3. Alan Jeffrey, Advanced Engineering Mathematics, Academic Press
4. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons
5. Gerald C.F and Wheatley P.O, Applied Numerical Analysis, Addison-Wesley Publishing Company

MODEL QUESTION PAPER

Reg.No. | | | | | | | | | |

B.E./B.Tech DEGREE EXAMINATIONS
SEMESTER –IV, APR./MAY.- 2020

SUB.CODE : CBSMAJ8T10
SUB.NAME : Mathematics-IV
Time: 3 Hours **Maximum: 100 Marks**

PART-A

Answer ALL Questions: **(10x2=20)**

1. Define a saddle point. (K2)
2. If $u = x^2$ and $v = y^2$, find $\frac{\partial(u,v)}{\partial(x,y)}$. (K1)
3. If $\phi(x,y,z) = x^2y + y^2x + z^2$, then find $\nabla\phi$ at the point (1,1,1). (K2)
4. Find the directional derivative of $4x^2z + xy^2$ at the point (1,-1,2) in the direction of the vector $2\bar{i} - \bar{j} + 3\bar{k}$ (K2)
5. Define recurrence relation (K2)
6. Define indicial equation on series solution when x=0 is a regular singularity (K1)
7. Write the Rodrigues's formula (K1)
8. Write the generating function of Legendre polynomials (K2)
9. What is the aim of design of experiments? (K2)
10. State the advantage and disadvantage of randomized block design. (K1)

Part - B

Answer ALL Questions: **5 x 16 = 80**

11. Find the dimensions of the rectangular box without top of maximum capacity with surface area 432 square metre. (K3)

(OR)

12. If $u = f(2x-3y, 3y-4z, 4z-2x)$, then prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$. (K6)

13. If $\vec{r}$ is the position vector of the point (x, y, z), prove that $\nabla^2 r^n = n(n+1)r^{n-2}$

and hence deduce $\nabla\left(\frac{1}{r}\right)$. (K4)

(OR)

14. Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region bounded by $x = 0, y = 0, x + y = 1$ (K4)

15. Solve in series the equation $\frac{d^2y}{dx^2} + xy = 0$ (K3)

(OR)

16. Prove that $J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{(3-x^2)}{x^2 \sin x} - \frac{3}{x} \cos x\right\}$.

(K6)

17. Prove that $(1-x^2)\, P_n'(x) = n[P_{n-1}(x) - x\, P_n(x)]$. (K3)

(OR)

18. State and prove the Orthogonal property on Legendre polynomials (K4)

19. The following data represent the no. of units of productions per day turned out by different workers using 4 different types of machines

	Machine A	B	C	D
I	44	38	47	36
II	46	40	52	43
III	34	36	44	32
IV	43	38	46	33
V	38	42	49	39

Test whether the five men differ with respect to mean productivity and test whether the mean productivity is the same for the four different machine type. (K6)

(OR)

20. The following is a Latin square of a design, when 4 varieties of seeds are being tested. Setup the analysis of variance table and state your conclusion. You may carry out suitable change of origin and scale.

A	105	B	95	C	125	D	115

C	115	D	125	A	105	B	105
D	115	C	95	B	105	A	115
B	95	A	135	D	95	C	115

(K5)

Blooms Taxonomy Level	Remembering (K1)	Understanding (K2)	Applying (K3)	Analyzing (K4)	Evaluating (K5)	Creating (K6)
Percentage	**4%**	**6%**	**27%**	**27%**	**9%**	**27%**

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PART - A

Unit – I

1. Define homogenous function. (K1)
2. State Euler's theorem (K2)
3. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ if $u = x^2y - \sin(xy)$ (K1)
4. If $z = f(x+ct) + g(x-ct)$ prove that, $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ (K2)
5. Find $\frac{dy}{dx}$ when $x^3 + y^3 = 3ax^2y$ (K1)
6. Find $\frac{dy}{dx}$ when, $x^y + y^x = c$. (K1)
7. If $z = u^2 + v^2$ and $u = at^2$, $v = 2at$ find $\frac{dz}{dt}$ (K2)
8. Find $\frac{du}{dt}$ given, $u = y^2 - 4ax$, $x = at^2$, $y = 2at$. (K2)
9. Write the working rule to find Maximum and Minimum value of f(x,y). (K1)
10. Define a saddle point. (K2)
11. If $u = x^2$ and $v = y^2$, find $\frac{\partial(u,v)}{\partial(x,y)}$. (K1)
12. If $u = xy$ and $v = x^2$, find $\frac{\partial(u,v)}{\partial(x,y)}$. (K1)
13. Find $\frac{du}{dx}$ if $u = x^2 y$, where $x^2 + xy + y^2 = 1$. (K2)
14. State the properties of Jacobian. (K1)
15. Write the Taylor's series expansion of $f(x,y)$ about (a,b). (K2)

UNIT - I

Maths Assignment 111891C106

1) Define Homogenous functions?

(a) An expression in x and y in which the sum of the indices of the variables x and y in each term in the same order is called Homogeneous function.

Eg: x^3+y^3, x^2+y^2 are Homogenous

2) State Eulers theorem?

If u is a homogenous function of degree n,

$$x \cdot \frac{du}{dx} + y\frac{du}{dy} = nu$$

where n = degree

3) If find $\frac{du}{dx}$ and $\frac{\partial u}{\partial y}$ if $u = x^2y - \sin xy$

$$u = x^2y - \sin xy$$

$$\frac{du}{dx} = 2xy - y\cos xy \Rightarrow 2xy - y\cos xy$$

$$\frac{\partial u}{\partial y} = x^2 - \cos xy \cdot x \Rightarrow x^2 - x\cos xy$$

4) If $z = f(x+ct) + g(x-ct)$ P.T $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$

To prove that,

Given $z = f(x+ct) + g(x-ct)$

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

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$$\frac{\partial z}{\partial t} = f'(x+ct)(c) + g'(x-ct)(-c)$$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+ct)c.c + g''(x-ct)(-c)(-c)$$

$$: f''(x+ct)c^2 + g''(x-ct)c^2$$

$$\frac{\partial z}{\partial x} : f'(x+ct) + g'(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2\left[f''(x+ct) + g''(x-ct)\right]$$

$$\therefore \frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Hence proved

5) Find $\frac{dy}{dx}$ when $x^3+y^3 = 3ax^2y$

Given $f = x^3+y^3-3ax^2y$

$$\frac{dy}{dx} = \frac{-\partial f}{\partial x} \Big/ \partial f/\partial y$$

$\partial f/\partial x = 3x^2+0-6axy$; $\frac{\partial f}{\partial y} = 3y^2-3ax^2$

$$= \frac{-(3x^2-6axy)}{(3y^2-3ax^2)}$$

$$= \frac{-x^2+2axy}{y^2-ax^2}$$

6) Find $\frac{dy}{dx}$ when $x^y + y^x = c$?

Sol: Given: $f = x^y + y^x - c = 0$

By implicit fn/.

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

$$\frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

$$\therefore \frac{-(yx^{y-1} + y^x \log y)}{(x^y \log x + xy^{x-1})}$$

$$= \frac{-yx^{y-1} - y^x \log y}{x^y \log x + xy^{x-1}}$$

3) If $z = u^2 + v^2$ and $u = at^2$, $v = 2at$ find $\frac{dz}{dt}$?

$$\frac{dz}{dt} = \frac{dz}{du} \cdot \frac{du}{dt} + \frac{dz}{dv} \cdot \frac{dv}{dt}$$

$$\frac{dz}{du} = 2u \qquad \frac{dz}{dv} = 2v$$

$$\frac{du}{dt} = 2at \qquad \frac{dv}{dt} = 2a$$

$$\frac{dz}{dt} = 2u(2at) + 2v(2a)$$

$$= 4uat + 4va$$

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$= 4a(ut+v)$

$= 4a((at^2)t+2at)$

$= 4a^2t(t^2+2)$

$\therefore \frac{dz}{dt} = 4a^2t(t^2+2)$ //.

8) Find $\frac{du}{dt}$ given $u = y^2 - 4ax$, $x = at^2$, $y = 2at$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Given $\Rightarrow x = at^2$, $y = 2at$

$\frac{\partial u}{\partial x} = -4a$ $\qquad \frac{\partial u}{\partial y} = 2y$

$\frac{dx}{dt} = 2at$ $\qquad \frac{dy}{dt} = 2a$

$$\frac{du}{dt} = -4a(2at) + 2y(2a)$$

$= -8a^2t + 4ay$

$= -8a^2t + 4a(2at)$

$= -8a^2t + 8a^2t$

$\therefore du/dt = 0$

9) Write the working rule to find maximum and minimum value of $f(x,y)$

A) * Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

* find x and y values and paired values

* calculate r, s, t values

* find $rt - s^2$

(i) If $rt - s^2 > 0$ and $r < 0 \Rightarrow$ maximum value

(ii) If $rt - s^2 > 0$ and $r > 0 \Rightarrow$ minimum value

(iii) If $rt - s^2 < 0$ and $r > 0, r < 0 \Rightarrow$ saddle point

(iv) If $rt - s^2 = 0$ then the method fails

10) Define saddle point

A) A value of function of two variables which is a maximum w.r.t one and a minimum with respect to other.

If $rt - s^2 < 0$ and $r > 0, r < 0$ then it is called as saddle point

11) If $u = x^2$ and $v = y^2$ find $\frac{\partial(u,v)}{\partial(x,y)} = ?$

A) Jacobians Matrix is $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

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$u = x^2$ and $v = y^2$

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial v}{\partial y} = 2y$ $\Rightarrow \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix}$

$\frac{\partial u}{\partial y} = 0$ $\quad \frac{\partial v}{\partial x} = 0$

$= 4xy$//.

12) If $u = xy$ and $v = x^2$ find $\frac{\partial(u,v)}{\partial(x,y)}$

A) Jacobians Matrix of $\frac{\partial(u,v)}{\partial(x,y)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$u = xy$ $\quad v = x^2$

$\frac{\partial u}{\partial x} = y$ $\quad \frac{\partial v}{\partial x} = 2x$ $\Rightarrow \begin{vmatrix} y & x \\ 2x & 0 \end{vmatrix}$

$\frac{\partial u}{\partial y} = x$ $\quad \frac{\partial v}{\partial y} = 0$

$= -2x^2$

13) Find $\frac{du}{dx}$ if $u = x^2 y$ where $x^2 + xy + y^2 = 1$

A) $\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$

$\frac{\partial u}{\partial x} = 2xy$, $\quad \frac{\partial u}{\partial y} = x^2$

$\frac{dy}{dx} = 2x + y + 2y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x - y}{2y}$

$$\frac{dy}{dx} = 2xy + x^2\left(\frac{-2x-y}{2y}\right)$$

$$\frac{dy}{dx} = \frac{4xy^2 - 2x^3 - x^2 y}{2y}$$

14) State properties of Jacobian?

A) Properties:

$$* \ \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

$$* \ \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

15) write the Taylors series expansion of $f(x,y)$ about (a,b)

A) Taylor series:

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[x f_x(a,b) + y f_y(a,b)\right] + \frac{1}{2!}\left[x^2 f_{xx}(a,b) + 2xy f_{xy}(a,b) + y^2 f_{yy}(a,b)\right] + \frac{1}{3!}\left[x^3 f_{xxx}(a,b) + 3x^2 y f_{xxy}(a,b) + 3y^2 x f_{yyx}(a,b) + y^3 f_{yyy}(a,b)\right]$$

PART – B

Unit – I

Differential Calculus

1. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$ (K4)

2. A rectangular box open at the top, is to have a volume of 32 cc. Find the dimensions of the box, that requires the least material for its construction. (K3)

3. A rectangular box open at the top is to have a given capacity K. Find the dimensions of the box requiring least material for its construction. (K3)

4. Find the dimensions of the rectangular box without top of maximum capacity with surface area 432 square metre. (K3)

5. Find the maximum and minimum value of $x^2 + y^2 + z^2$ subject to the condition $x + y + z = 3a$. (K5)

6. Find the extreme values of the function, $f(x, y) = x^3 + y^3 - 3x - 12y + 20$. (K5)

7 Find the extreme values of the function, $f(x, y) = x^3y^2(1 - x - y)$. (K5)

8. Find the three positive numbers such that their sum is a constant ‘a’ and their product is maximum. (K6)

9. Find the extreme values of the function $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$. (K4)

10. Find the extreme values of the function $f(x, y) = x^3 + y^3 - 12x - 3y + 20$. (K4)

11 If $u = f(x-y, y-z, z-x),$ then prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$ (K6)

12 If $u = f(2x-3y, 3y-4z, 4z-2x),$ then prove that $\frac{1}{2}\frac{\partial u}{\partial x}+\frac{1}{3}\frac{\partial u}{\partial y}+\frac{1}{4}\frac{\partial u}{\partial z}=0.$ (K6)

13. If z is a function $f(x,y)$, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, then prove that

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$$

(K6)

14 If $u = f(x,y)$, where $x = r\cos\theta$, $y = \sin\theta,$ then Prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

(K6)

15 If $Z = f(u,v)$, where $u = lx + my,\ v = ly = mx$ then prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

(K6)

16. Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$, $u = e^x \cos y,\ v = e^x \sin y$ and that f is a function of u and v and also of x and y.

17. If $u = u(x, y)$ and $x = e^r \cos\theta$, $y = e^r \sin\theta$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 z}{\partial r^2} + \frac{\partial^2 z}{\partial \theta^2}\right)$

(K6)

18. Obtain terms upto the third degree in the Taylor's series expansion of $e^x \sin y$ around the point $\left(1, \frac{\pi}{2}\right)$.

(K5)

19 Find the Taylor's series expansion of e^{xy} near the point $(1,1)$ upto the second degree terms.

(K4)

UNIT - I

DIFFERENTIAL CALCULAS

PART-B

(2) A rectangular box open at the top is to have a volume of 32cc. Find the dimensions of the box, that requires the least material for its construction.

Sol:- Let x, y, z be the length, breadth and height of the box.

$$f = xy + 2yz + 2zx \quad \text{---} (1)$$

$$\phi = xyz - 32 \quad \text{---} (2)$$

Let the auxilary can be $g = f + \lambda\phi$

$$g = xy + 2yz + 2zx + \lambda(xyz - 32)$$

$$g_x = y + 0 + 2z + \lambda(yz)$$

$$g_y = x + 2z + 0 + \lambda(xz)$$

$$g_z = 0 + 2y + 2x + \lambda(yx)$$

Find the stationary point $g_x = 0, g_y = 0, g_z = 0$

$$y + 2z + \lambda(yz) = 0$$

$$\lambda(yz) = -(2z + y)$$

$$\lambda = \frac{-(2z+y)}{yz}$$

$$x + 2z + \lambda(xz) = 0$$

$$\lambda = \frac{-(2z+x)}{xz}$$

$$2y + 2x + \lambda(yx) = 0$$

$$\lambda = \frac{-(2x+2y)}{yx}$$

$$\lambda = \frac{-2z}{yz} = \frac{-y}{yz}$$

$$\boxed{\lambda = \frac{-2}{y} - \frac{1}{z}} \quad \text{---} (3)$$

$$\lambda = \frac{-2z}{xz} = \frac{x}{xz}$$

$$\boxed{\lambda = \frac{-2}{x} - \frac{1}{z}}$$

$$\lambda = \frac{-2x}{yx} - \frac{2y}{yx}$$

$$\boxed{\lambda = -\frac{2}{y} - \frac{2}{x}}$$

$$-\lambda = \frac{2}{y} + \frac{1}{z}$$

$$-\lambda = \frac{2}{x} + \frac{1}{z}$$

$$-\lambda = \frac{2}{y} + \frac{2}{x}$$

Solving ③, ④ & ⑤

$③ \Rightarrow \frac{1}{z} + \frac{2}{y} = -\lambda$

$④ \Rightarrow \frac{1}{z} + \frac{2}{z} = -\lambda$

$\frac{2}{y} - \frac{2}{x} = 0$

$\frac{2}{y} = \frac{2}{x} \Rightarrow \boxed{x = y}$

sub $y = x$ in eqn ⑤ $\Rightarrow -\lambda = \frac{2}{x} + \frac{2}{y}$

$= \frac{2}{x} + \frac{2}{x}$

$\boxed{x = \frac{-4}{\lambda}}$

Illy $\Rightarrow y = \frac{-4}{\lambda}$

solving ④ & ⑤ we get $y = 2z$

$\boxed{\therefore z = \frac{-2}{\lambda}}$

Sub x, y & z in eqn ②

$\phi: \left(\frac{-4}{\lambda}\right)\left(\frac{-4}{\lambda}\right)\left(\frac{-2}{\lambda}\right) = 32$

$\Rightarrow = \frac{-32}{\lambda^3} = 32$

$\frac{-32}{\lambda^3} = 32$

$\lambda^3 = -1$

$\boxed{\lambda = -1}$

When $\lambda = -1$ then $x = 4$, $y = 4$, $z = 2$

The rectangular box length is 4cm, breadth is 4cm & height 4cm.

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② A rectangular box open at the top is to have a volume of capacity of k. Find the dimensions of the box, that requires the least material for its construction.

Sol: Let x, y, z be the length, breadth and height of the box.

$$f = xy + 2yz + 2zx \quad \text{—} (1)$$

$$\phi = xyz - k \quad \text{—} (2)$$

Let the auxilary can be $g = f + \lambda\phi$

$$g = xy + 2yz + 2zx + \lambda(xyz - k)$$

$$g_y = x + 2z + 0 + \lambda(xz)$$

$$g_x = y + 0 + 2z + \lambda(yz)$$

$$g_z = 0 + 2y + 2x + \lambda(xy)$$

Find the stationary point $g_x = 0; g_y = 0; g_z = 0$

$$y + 2z + \lambda(yz) = 0$$

$$\lambda = \frac{-(2z + y)}{yz}$$

$$-\lambda = \frac{2z}{yz} + \frac{y}{yz}$$

$$\boxed{-\lambda = \frac{2}{y} + \frac{1}{z}} \quad \text{—} (3)$$

$$x + 2z + \lambda(xz) = 0$$

$$\lambda = \frac{-(2z + x)}{xz}$$

$$-\lambda = \frac{2z}{xz} + \frac{x}{xz}$$

$$\boxed{-\lambda = \frac{2}{x} + \frac{1}{z}} \quad \text{—} (4)$$

$$2y + 2x + \lambda(xy) = 0$$

$$\lambda = \frac{-(2y + 2x)}{xy}$$

$$-\lambda = \frac{2y}{xy} + \frac{2x}{xy}$$

$$\boxed{-\lambda = \frac{2}{x} + \frac{2}{y}} \quad \text{—} (5)$$

solving (3), (4) & (5)

$$(3) \Rightarrow \frac{1}{z} + \frac{2}{y} = -\lambda$$

$$\frac{1}{z} + \frac{2}{x} = -\lambda$$

$$\frac{2}{y} - \frac{2}{x} = 0 \Rightarrow \boxed{y = x}$$

Sub $y=x$ in eqn ⑤ $\Rightarrow -\lambda = \frac{2}{x} + \frac{2}{y}$

$$-\lambda = \frac{2}{x} + \frac{2}{x}$$

$$\lambda = \frac{4}{x} \quad \boxed{x = \frac{-4}{\lambda}}$$

lly $y = \frac{-4}{\lambda}$

solving eqn ④ & ⑥ we get $y = 2z$

$$\boxed{\therefore z = \frac{-2}{\lambda}}$$

Sub x, y & z in Eqn ②

$$\phi = \left(\frac{-4}{\lambda}\right)\left(\frac{-4}{\lambda}\right)\left(\frac{-2}{\lambda}\right) - k$$

$$= \frac{-32}{\lambda^3} - k$$

$$k = \frac{-32}{\lambda^3}$$

$$\lambda^3 k = -32$$

$$x = \frac{-4}{-\left(\frac{32}{k}\right)^{1/3}} = \frac{4(k)^{1/3}}{(32)^{1/3}} = (2k)^{1/3}$$

$$y = \frac{-4}{-\left(\frac{32}{k}\right)^{1/3}} = \frac{4(k)^{1/3}}{(32)^{1/3}} = (2k)^{1/3}$$

$$z = \frac{-2}{-\left(\frac{32}{k}\right)^{1/3}} = \left(\frac{k}{4}\right)^{1/3}$$

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4. Find the dimensions of a rectangular box without top of maximum capacity with surface area 432 sq m

Sol Given Let x, y, z be the length, breadth and height of the box ~~tool~~ respectively.

Given, surface area $= 432$

$$S = xy + 2yz + 2zx = 432 = \phi$$

$$V = xyz = f$$

Let $g = f + \lambda\phi$

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$g = xyz + \lambda(xy + 2yz + 2zx - 432)$

$\frac{\partial g}{\partial x} = yz + (y+2z)\lambda$ — ①

$\frac{\partial g}{\partial y} = xy - xz + (x+2z)\lambda$ — ②

$\frac{\partial g}{\partial z} = xy + \lambda(2y+2x)$ — ③

① ⇒ $yz + (y+2z)\lambda = 0$

$y\lambda + 2z\lambda = -yz$

divide with yz

$\cancel{\frac{\lambda}{z} + 2y}$ $\frac{\lambda}{z} + \frac{2\lambda}{y} = -1$ — ④

② ⇒ $x\lambda + 2z\lambda + xz = 0$

$x\lambda + 2z\lambda = -xz$

divide with xz

$\frac{\lambda}{z} + \frac{2\lambda}{x} = -1$ — ⑤

③ ⇒ $2y\lambda + 2x\lambda = -xy$

divide with xy

$\frac{2\lambda}{x} + \frac{2\lambda}{y} = -1$ — ⑥

Solve ④ & ⑤

eqn ④ = eqn ⑤

$\frac{2\lambda}{y} = \frac{2\lambda}{x} \Rightarrow \boxed{y = x}$

Put $y = x$ in eqn ⑥ we get

$\frac{2\lambda}{x} + \frac{2\lambda}{x} = -1$

$\frac{4\lambda}{x} = -1$

$-x = 4\lambda$ lly $y = -4\lambda$

$\boxed{x = -4\lambda}$

Put y value in eqn ④ we get

$\frac{\lambda}{z} + \frac{2\lambda}{-4\lambda} = -1$

$\frac{\lambda}{z} + \frac{1}{2} = -1$

$\frac{\lambda}{2} = -1+\frac{1}{2} \Rightarrow \frac{\lambda}{2} = \frac{1}{2}$

$\boxed{2\lambda = z}$

Sub in ③ $\Rightarrow$ $xy + 2yz + 2zx = 432$

$(-4\lambda)(-4\lambda) + 2(-4\lambda)(-2\lambda) + 2(-2\lambda)(-4\lambda) = 432$

$16\lambda^2 + 16\lambda^2 + 16\lambda^2 = 432$

$\lambda^2 = 432/48 = 9$

$\boxed{\lambda = \pm 3}$

$x = -4\lambda$	$y = -4\lambda$	$z = 2\lambda$
$x = -4(3)$	$y = -4(3)$	$= 2(3)$
$x = -12$	$y = -12$	$\boxed{z = 6}$
(or)	(or)	
$\boxed{x = 12}$	$\boxed{y = 12}$	

The length, breadth and height of a rectangular box should always positive

$\therefore$ The length $= 12$cm, breadth $= 12$, height $= 6$cm

---

⑤ Find the maximum and minimum value of $x^2+y^2+z^2$ subject to the condition $x+y+z = 3a$

Sol:- $f = x^2+y^2+z^2$ — ①

$\phi = x+y+z-3a$ — ②

$g = f + \lambda\phi$

$g = x^2+y^2+z^2+\lambda(x+y+z-3a)$

$g_x = 2x+\lambda$

$g_y = 2y+\lambda$

$g_z = 2z+\lambda$

Find the stationary point $g_x = 0$; $g_y = 0$; $g_z = 0$

$2x+\lambda = 0$

$\lambda = -2x$ $\boxed{-\lambda = 2x}$ — ③

$2y+\lambda=0$

$\lambda=-2y \Rightarrow \boxed{-\lambda=2y}$ — ④

$2z+\lambda=0$

$\boxed{\lambda=-2z} \Rightarrow \boxed{-\lambda=2z}$ — ⑤

Solving ③, ④, ⑤

$x=-\frac{\lambda}{2}$; $y=-\frac{\lambda}{2}$; $z=-\frac{\lambda}{2}$

Sub x, y & z in eqn ②

$x+y+z-3a=0$

$\left(-\frac{\lambda}{2}\right)+\left(-\frac{\lambda}{2}\right)+\left(-\frac{\lambda}{2}\right)-3a=0$

$-\frac{3\lambda}{2}-3a=0$

$-\frac{3\lambda}{2}=3a$

$2a=-\lambda$ $\boxed{\lambda=-2a}$

Sub λ in x, y, z

$x=-\frac{(-2a)}{2}$ ||ly $y=-\frac{(-2a)}{2}$ $z=-\frac{(-2a)}{2}$

$\boxed{x=a}$ $\boxed{y=a}$ $\boxed{z=a}$

∴ The max value of $f=a^2+a^2+a^2$

$=3a^2$

---

⑥ Find the extreme values of the function, $f(x,y)=x^3+y^3-3x+12y+20$

Sol:-

$f(x,y)=x^3+y^3-3x-12y+20$

$\frac{\partial f}{\partial x}=3x^2-3$ — ①

$\frac{\partial f}{\partial y}=3y^2-12$ — ②

$\frac{\partial f}{\partial x}=0 \Rightarrow (3x^2-3=0)\times 4$

$12x^2-12=0$ — ③

$\frac{\partial f}{\partial y}=0 \Rightarrow 3y^2-12=0$ — ④

Solve eqn ③ & ④

$$12x^2 - 12 = 0$$
$$3x^2 - 12 = 0$$

$$12x^2 - 3y^2 = 0$$
$$12x^2 = 3y^2$$
$$y^2 = 4x^2$$
$$y^2 = (2x)^2$$
$$\boxed{y = 2x}$$

Sub $y = 2x$ in eqn ②

$$3(2x)^2 - 12 = 0$$
$$12x^2 - 12 = 0$$
$$(x^2 - 1) = 0$$
$$x^2 = 1$$
$$\boxed{x = \pm 1}$$

$x = 1,\ y = 2$

$x = -1,\ y = -2$

The paired values are $(1,2)$; $(1,-2)$; $(-1,2)$ $(-1,-2)$

$$r = \frac{\partial^2 f}{\partial x} = 6x$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

Paired values	r	t	s	s^2	rt	$rt - s^2$	result	Value
(1,2)	6	12	0	0	72	72	$rt - s^2 > 0$ $r > 0$ min value	$f(1,2) = 2$
(1,-2)	6	-12	0	0	-72	-72	$rt - s^2 < 0$ $r > 0$	$f(1,-2) = 34$
(-1,2)	-6	12	0	0	-72	-72	$rt - s^2 < 0$ $r < 0$	$f(-1,2) = 6$
(-1,-2)	-6	-12	0	0	72	72	$rt - s^2 > 0$ $r < 0$ max value	$f(-1,-2) = 38$

$f(x,y) = x^3+y^3-3x-12y+20$

$f(1,2) = 1+8-3-24+20$

$= 2$

$f(1,-2) = 1-8-3+24+20$

$= 34$

$f(-1,2) = -1+8+3-24+20$

$= 6$

$f(-1,-2) = -1-8+3+24+20$

$= 38$

$\therefore$ The max value is 38 at $f(-1,2)$

The min value is 2 at $f(1,2)$

---

(7) Find the extreme value of the function $f(x,y) = x^3y^2(1-x-y)$

Sol:- $f(x,y) = x^3y^2(1-x-y) \Rightarrow x^3y^2 - x^4y^2 - x^3y^3$

Step 1:-

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$$

Step 2:-

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0$$

$$x^3y[2-2x-3y] = 0$$

$$x^2y^2[3-4x-3y] = 0$$

$$3-4x-3y = 0 \Rightarrow x=0, y=0$$

$$4x+3y=3 \quad -(1)$$

$$x^3y[2-2x-3y] = 0$$

$$2-2x-3y = 0 \Rightarrow x=0, y=0$$

$$2x+3y=2 \quad -(2)$$

Solve (1) & (2) $\quad 4x+3y=3$

$2x+3y=2$

$2x=1$

$\boxed{x=1/2}$

Sub $x=1/2$ in eqn ①

$4(1/2)+3y=3$

$2+3y=3$

$3y=1$

$\boxed{y=1/3}$

The stationary points are $(0,0), (0,1/3), (1/2,0), (1/2,1/3)$

$$r=\frac{\partial^2 f}{\partial x^2}=6xy^2-12x^2y^2-6xy^3$$

$$s=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=6x^2y-8x^3y-9x^2y^2$$

$$t=\frac{\partial^2 f}{\partial y^2}=2x^3-2x^4-6x^3y$$

Pair of Values	r	t	rt	s	s^2	$rt-s^2$	result	Value
$(0,0)$	0	0	0	0	0	0	fails $rt-s^2=0$	fails
$(0,1/3)$	0	0	0	0	0	0	fails $rt-s^2=0$	fails
$(1/2,0)$	0	$1/3$	0	0	0	0	fails $rt-s^2=0$	fails
$(1/2,1/3)$	$-\frac{1}{9}$	$-\frac{1}{8}$	$\frac{1}{72}$	$-\frac{1}{12}$	$\frac{1}{144}$	$\frac{1}{144}$	$rt-s^2>0$ $r<0$ max	0.0023

$$r=6\left(\tfrac{1}{2}\right)\left(\tfrac{1}{9}\right)-12\left(\tfrac{1}{4}\right)\left(\tfrac{1}{4}\right)-6\left(\tfrac{1}{2}\right)\left(\tfrac{1}{27}\right)$$

$$=\frac{1}{3}-\frac{1}{3}-\frac{1}{9}$$

$$\boxed{r=\frac{-1}{9}}$$

$$t=2(1/8)-2(1/16)-6(1/8)(1/3)$$

$$=\frac{1}{4}-\frac{1}{8}-\frac{1}{4}=-1/8$$

$$s=6\left(\tfrac{1}{4}\right)\left(\tfrac{1}{3}\right)-8\left(\tfrac{1}{8}\right)\left(\tfrac{1}{3}\right)-9\left(\tfrac{1}{4}\right)\left(\tfrac{1}{9}\right)$$

$$=1/2-1/3-1/4=1/4-1/3 \quad \boxed{s=-1/12}$$

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$f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

$= \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2 - \left(\frac{1}{2}\right)^4\left(\frac{1}{3}\right)^2 - \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^3$

$= \left(\frac{1}{8}\right)\left(\frac{1}{9}\right) - \left(\frac{1}{16}\right)\left(\frac{1}{9}\right) - \left(\frac{1}{8}\right)\left(\frac{1}{27}\right)$

$= \frac{1}{72}\left[1 - \frac{1}{2} - \frac{1}{3}\right)$

$= \frac{1}{152}$

---

(9) Find the extreme values of the function

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

Sol: $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ — (1) $\quad \frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$ — (2)

$\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$ — (3)

$\frac{\partial f}{\partial x} = 0 \Rightarrow 4(x^3 - x + y) = 0$

$x^3 - x + y = 0$ — (4)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 4(y^3 + x - y) = 0$

$y^3 + x - y = 0$ — (5)

Solving (4) & (5) we get

$x^3 - x + y = 0$

$y^3 + x - y = 0$

$x^3 + y^3 = 0 \qquad x^3 = -y^3$

$x = -y$

$y = -x$

Solve $y = -x$ in (2)

$4x^3 - 4x + 4(-x) = 0$

$4x^3 - 8x = 0 \;;\; 4(x^3 - 2x) = 0$

$x(x^2 - 2) = 0$

$x = 0,\; x^2 = 2$

$x = \pm\sqrt{2}$

$x = 0 \;;\; y = 0 \;;\; x = \sqrt{2} \Rightarrow y = -\sqrt{2}$

$x = -\sqrt{2} \Rightarrow y = \sqrt{2}$

Paired values are $(0,\sqrt{2})$ $(\sqrt{2},0)$ $(-\sqrt{2},0)$

$(0,-\sqrt{2})$ $(0,0)$ $(\sqrt{2},-\sqrt{2})$

$(\sqrt{2},\sqrt{2})$ $(-\sqrt{2},\sqrt{2})$ $(-\sqrt{2},0)$

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2-4$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 4$$

$$t = \frac{\partial^2 f}{\partial y^2} = 12y^2-4$$

Paired values	r	t	rt	s	s^2	$rt-s^2$	result	Value
$(0,0)$	-4	-4	16	4	16	0	$rt-s^2=0$ fails	fails
$(\sqrt{2},-\sqrt{2})$	20	20	400	4	16	384	$rt-s^2>0$	$f(\sqrt{2},-\sqrt{2})=-8$
$(\sqrt{2},\sqrt{2})$	20	20	400	4	16	384	$rt-s^2>0$	$f(\sqrt{2},\sqrt{2})=8$
$(\sqrt{2},0)$	20	-4	-80	4	16	-90	$rt-s^2<0$	$f(\sqrt{2},0)=0$ saddle point
$(-\sqrt{2},\sqrt{2})$	20	20	400	4	16	384	$rt-s^2>0$	$f(-\sqrt{2},\sqrt{2})=-8$
$(0,-\sqrt{2})$	-4	20	-80	4	16	-96	$rt-s^2<0$	$f(0,-\sqrt{2})=0$ saddle point
$(0,\sqrt{2})$	-4	20	-80	4	16	-96	$rt-s^2<0$	$f(0,\sqrt{2})=0$ saddle point

(10) Find the extreme values of the function

$$f(x,y) = x^3+y^3-12x-3y+20$$

Sol:- $f(x,y) = x^3+y^3-12x-3y+20 \quad -(1)$

$$\frac{\partial f}{\partial x} = 3x^2-12=0 \quad -(2)$$

$$\frac{\partial f}{\partial y} = 3y^2-3=0 \quad -(3)\times 4$$

$$=12y^2-12=0$$

Solve (2) & (3) $\Rightarrow$ $3x^2-12=0$

$$12y^2-12=0$$

$$\overline{3x^2-12y^2=0}$$

$3x^2 = 12y^2$

$\boxed{x = 2y}$

Sub $x=2y$ in eqn (3)

$12y^2 - 12 = 0 \Rightarrow 12y^2 = 12$

$y^2 = 1$

$\boxed{y = \pm 1}$

Sub $x = 2y$ in eq (3)

$12y^2 - 12 = 0 \Rightarrow 12y^2 = 12$

$\boxed{y = \pm 1}$

$y = 1 \Rightarrow x = 2$

$y = -1 \Rightarrow x = -2$

Paired values :- $(2,1), (2,-1), (-2,1), (-2,-1)$

$r = \frac{\partial^2 f}{\partial x^2} = 6x$

$s = \frac{\partial f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$

$t = \frac{\partial^2 f}{\partial y^2} = 6y$

Paired value	r	t	rt	s	s^2	$rt - s^2$	result	Value
(2,1)	12	6	72	0	0	72	$rt - s^2 > 0$ $r > 0$	min value
(2,-1)	12	-6	-72	0	0	-72	$rt - s^2 < 0$ $r > 0$	
(-2,1)	-12	6	-72	0	0	-72	$rt - s^2 < 0$ $r < 0$	
(-2,-1)	-12	-6	72	0	0	72	$rt - s^2 > 0$ $r < 0$	max value

$f(2,1) = x^3 + y^3 - 12x - 3y + 20$

$= 8 + 1 - 24 - 3 + 20$

$f(2,1) = 2$ min value

$f(-2,-1) = (-2)^3 + (-1)^3 + 12(-2) - 3(-1) + 20$

$= -8 + (-1) + 24 + 3 + 20$

$= 38$ max value

(11) If $u=f(x-y,y-z,z-x)$ then prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

Let $x-y=r$, $y-z=s$; $z-x=t$

$$u=f(r,s,t)$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}(1)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}(-1)$$

$$=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t} \quad -(1)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$=\frac{\partial u}{\partial r}(-1)+\frac{\partial u}{\partial s}(-1)+\frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial y}=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s} \quad -(2)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}(-1)+\frac{\partial u}{\partial t}(1)$$

$$=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t} \quad -(3)$$

(1)+(2)+(3) =)

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}+\frac{\partial u}{\partial s}-\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

(12) If $w=f(2x-3y,3y-4z,4z-2x)$ then prove that

$$\frac{1}{2}\frac{\partial w}{\partial x}+\frac{1}{3}\frac{\partial w}{\partial y}+\frac{1}{4}\frac{\partial w}{\partial z}=0$$

Sol:-

$$w=f(2x-3y,3y-4z,4z-2x)$$

Let $2x-3y=r$	$3y-4z=s$	$4z-2x=t$
$\frac{\partial r}{\partial x}=(2)$	$\frac{\partial s}{\partial x}=0$	$\frac{\partial t}{\partial x}=-2$
$\frac{\partial r}{\partial y}=-3$	$\frac{\partial s}{\partial y}=3$	$\frac{\partial t}{\partial y}=0$
$\frac{\partial r}{\partial z}=0$	$\frac{\partial s}{\partial z}=-4$	$\frac{\partial t}{\partial z}=4$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial w}{\partial r}(2) + \frac{\partial w}{\partial s}(0) + \frac{\partial w}{\partial t}(-2)$$

$$= 2\left[\frac{\partial w}{\partial r} - \frac{\partial w}{\partial t}\right]$$

$$\frac{1}{2}\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} - \frac{\partial w}{\partial t} \quad -①$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}(-3) + \frac{\partial w}{\partial s}(3) + \frac{\partial u}{\partial t}(0)$$

$$= 3\left[-\frac{\partial w}{\partial r} + \frac{\partial w}{\partial s}\right]$$

$$\frac{1}{3}\frac{\partial u}{\partial y} = -\frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} \quad -②$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-4) + \frac{\partial u}{\partial t}(4)$$

$$\frac{\partial u}{\partial z} = 4\left[-\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}\right]$$

$$\frac{1}{4}\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \quad -③$$

①+②+③

$$\Rightarrow \frac{1}{2}\frac{\partial w}{\partial x} + \frac{1}{3}\frac{\partial w}{\partial y} + \frac{1}{4}\frac{\partial w}{\partial z} = \cancel{\frac{\partial w}{\partial r}} - \cancel{\frac{\partial w}{\partial t}} - \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}} - \cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}}$$

$$\frac{1}{2}\frac{\partial w}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$$

13\. If z is a function $f(x,y)$, where $x = e^{u} + e^{-v}$ and $y = e^{-u} - e^{v}$ then prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Sol

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^{u}) + \frac{\partial z}{\partial y}(-e^{-u}) \quad -① $$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}) \quad -②$$

① - ② =>

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^{u}) - \frac{\partial z}{\partial y}(e^{-u}) + \frac{\partial z}{\partial x} e^{-v} + \frac{\partial z}{\partial y} e^{v}$$

$$= \frac{\partial z}{\partial x}\left[e^{u} + e^{-v}\right] + \frac{\partial z}{\partial y}\left[-e^{-u} + e^{v}\right]$$

$$= \frac{\partial z}{\partial x}(x) - \frac{\partial z}{\partial y}\left[e^{-u} - e^{v}\right]$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \cdot \frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} \text{ //}$$

---

⑭ If $u = f(x,y)$, where $x = r\cos\theta$, $y = \sin\theta$, then prove that,

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

Sol:-

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = r(-\sin\theta) = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad -①$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\cos\theta \quad -②$$

$$①^2 \Rightarrow \left(\frac{\partial u}{\partial r}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta + 2\frac{\partial u}{\partial x}\cos\theta\frac{\partial u}{\partial y}\sin\theta \quad -③$$

$$②^2 \Rightarrow \left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2(r^2\sin^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2 r^2\cos^2\theta + 2\frac{\partial u}{\partial x}(-r\sin\theta)\frac{\partial u}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial u}{\partial \theta}\right)^2 = r^2\left[\left(\frac{\partial u}{\partial x}\right)^2(r^2\sin^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2 r^2\cos^2\theta + 2\frac{\partial u}{\partial x}(-r\sin\theta)\frac{\partial u}{\partial y}(r\cos\theta)\right]$$

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$$\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left[\left(\frac{\partial u}{\partial x}\right)^2 \sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 \cos^2\theta - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta\right] - (4)$$

(3) + (4) =>

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left[\left(\frac{\partial u}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 \sin^2\theta + \cancel{2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\cos\theta\sin\theta}\right]$$

$$+ \left(\frac{\partial u}{\partial x}\right)^2 \sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 \cos^2\theta \cancel{-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta}\Big]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 (\cos^2\theta + \sin^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2 (\sin^2\theta + \cos^2\theta)$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 \,//$$

15) If $z = f(u, v)$ where $u = lx + my$; $v = ly - mx$ then prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right]$

Sol:- Given $u = lx + my$; $v = ly - mx$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(l) + \frac{\partial z}{\partial v}\cdot(-m)$$

$$\frac{\partial z}{\partial x} = l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v} - (1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(m) + \frac{\partial z}{\partial v}(l)$$

$$= m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v} - (2)$$

$$(1)^2 \Rightarrow \frac{\partial^2 z}{\partial x^2} = l^2\frac{\partial^2 z}{\partial u^2} + m^2\frac{\partial^2 z}{\partial v^2} - 2lm\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

$$(2)^2 \Rightarrow \frac{\partial^2 z}{\partial y^2} = m^2\frac{\partial^2 z}{\partial u^2} + l^2\frac{\partial^2 z}{\partial v^2} + 2lm\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

$$(1)^2 + (2)^2 = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = l^2\frac{\partial^2 z}{\partial u^2} + m^2\frac{\partial^2 z}{\partial v^2} - 2lm\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + m^2\frac{\partial^2 z}{\partial u^2}$$

$$+ l^2\frac{\partial^2 z}{\partial v^2} + \cancel{2lm\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}}$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(l^2+m^2)\left[\frac{\partial^2 z}{\partial u^2}\right]+(l^2+m^2)\left[\frac{\partial^2 z}{\partial v^2}\right]$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=l^2+m^2\left[\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right]$$

16) prove that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(u^2+v^2)\left[\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right]$; $u=e^x\cos y$, $v=e^x\sin y$, f is a function of u and v and also of x & y.

Sol,-

Given, $u=e^x\cos y$, $v=e^x\sin y$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}(e^x\cos y)+\frac{\partial z}{\partial v}(e^x\sin y) \quad \text{—①}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}(-e^x\sin y)+\frac{\partial z}{\partial v}(e^x\cos y) \quad \text{—②}$$

$①^2 \Rightarrow \frac{\partial^2 z}{\partial x^2}=(e^x\cos y)^2\frac{\partial^2 z}{\partial u^2}+(e^x\sin y)^2\frac{\partial^2 z}{\partial v^2}+2e^x\cos y\, e^x\sin y\,\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$ —③

$②^2 \Rightarrow \frac{\partial^2 z}{\partial y^2}=(e^x\sin y)^2\frac{\partial^2 z}{\partial u^2}+(e^x\cos y)^2\frac{\partial^2 z}{\partial v^2}-2e^x\cos y\, e^x\sin y\,\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$ —④

$③ \Rightarrow \frac{\partial^2 z}{\partial x^2}=u^2\frac{\partial^2 z}{\partial u^2}+v^2\frac{\partial^2 z}{\partial v^2}+2e^x\cos y\, e^x\sin y\,\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}$

$④ \Rightarrow \frac{\partial^2 z}{\partial y^2}=v^2\frac{\partial^2 z}{\partial u^2}+u^2\frac{\partial^2 z}{\partial v^2}-2e^x\cos y\, e^x\sin y\,\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}$

$③+④ \Rightarrow \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left[u^2\frac{\partial^2 z}{\partial u^2}+v^2\frac{\partial^2 z}{\partial v^2}+2e^x\cos y\, e^x\sin y\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}\right]+$

$\left[v^2\frac{\partial^2 z}{\partial u^2}+u^2\frac{\partial^2 z}{\partial v^2}-2e^x\cos y\, e^x\sin y\,\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}\right]$

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$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(u^2+v^2)\left(\frac{\partial^2 z}{\partial u^2}\right)+(u^2+v^2)\left(\frac{\partial^2 z}{\partial v^2}\right)$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(u^2+v^2)\left[\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right] //$$

17) If $z=u(x,y)$ and $x=e^r\cos\theta$, $y=e^r\sin\theta$. show that

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=e^{-2r}\left[\frac{\partial^2 z}{\partial r^2}+\frac{\partial^2 z}{\partial \theta^2}\right]$$

Sol:- Given $x=e^r\cos\theta$, $y=e^r\sin\theta$

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}(e^r\cos\theta)+\frac{\partial z}{\partial y}(e^r\sin\theta) \quad -①$$

$$\frac{\partial^2 z}{\partial r^2}=(e^r\cos\theta)^2\frac{\partial^2 z}{\partial x^2}+(e^r\sin\theta)^2\frac{\partial^2 z}{\partial y^2}+2e^r\cos\theta e^r\sin\theta\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} \quad -②$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\cdot(-e^r\sin\theta)+\frac{\partial z}{\partial y}(e^r\cos\theta) \quad -③$$

$$\frac{\partial^2 z}{\partial \theta^2}=(e^r\sin\theta)^2\frac{\partial^2 z}{\partial x^2}+(e^r\cos\theta)^2\frac{\partial^2 z}{\partial y^2}-2e^r\sin\theta e^r\cos\theta\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} \quad -④$$

② + ④ =)

$$\frac{\partial^2 z}{\partial r^2}+\frac{\partial^2 z}{\partial \theta^2}=\frac{\partial^2 z}{\partial x^2}\left[e^{2r}(\sin^2\theta+\cos^2\theta)\right]+\frac{\partial^2 z}{\partial y^2}\left[e^{2r}(\sin^2\theta+\cos^2\theta)\right]$$
$$+\cancel{2e^{2r}\sin\theta\cos\theta\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}}-\cancel{2e^{2r}\cos\theta\sin\theta\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}}$$

$$\frac{\partial^2 z}{\partial r^2}+\frac{\partial^2 z}{\partial \theta^2}=e^{2r}\left[\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right]$$

$$\frac{1}{e^{2r}}\left[\frac{\partial^2 z}{\partial r^2}+\frac{\partial^2 z}{\partial \theta^2}\right]=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=e^{-2r}\left[\frac{\partial^2 z}{\partial r^2}+\frac{\partial^2 z}{\partial \theta^2}\right] //$$

① Find the minimum value of $x^2+y^2+z^2$ subject to the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

Sol

$$f=x^2+y^2+z^2 \quad -① $$

$$\phi=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1 \quad -②$$

Let the auxilary equation be

$$g=f+\lambda\phi$$

$$=x^2+y^2+z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

$$g_x=2x+\left(\frac{-\lambda}{x^2}\right)=2x-\frac{\lambda}{x^2}$$

$$g_y=2y-\frac{\lambda}{y^2}$$

$$g_z=2z-\frac{\lambda}{z^2}$$

to find stationary points

$g_x=0$ $g_y=0$ $g_z=0$

$2x-\frac{\lambda}{x^2}=0$ $2y-\frac{\lambda}{y^2}=0$ $2z-\frac{\lambda}{z^2}=0$

$\lambda=2x^3$ $\lambda=2y^3$ $\lambda=2z^3$

$$\Rightarrow x=y=z=\left(\frac{\lambda}{2}\right)^{1/3}$$

Put x,y,z values in ②

$$\frac{1}{\left(\frac{\lambda}{2}\right)^{1/3}}+\frac{1}{(\lambda/2)^{1/3}}+\frac{1}{(\lambda/2)^{1/3}}-1=0$$

$$\frac{3}{(\lambda/2)^{1/3}}=1$$

$$\left(\frac{\lambda}{2}\right)^{1/3}=3$$

$$\therefore x=y=z=(\lambda/2)^{1/3}=3$$

Sub x,y,z in eq^n ①

minimum value of f

$$f=x^2+y^2+z^2$$

$$=3^2+3^2+3^2$$

$$\boxed{f=27}$$

(6) Find the three positive numbers such that their sum is a constant "a" and their product is maximum.

Sol:- The three positive numbers sum is a product of its maximum.

Let x, y, z are three positive numbers

$x+y+z=a$; $f(x,y)=xyz$

$z=a-(x+y)$

$z=a-x-y$

$\therefore f(x,y)=xyz$

$=xy(a-x-y)$

$=axy-x^2y-xy^2$

The product of 3 numbers is max then

$f_x=f_y=0$

$f_x=ay-2xy-y^2=0$

$y(a-2x-y)=0$

$y=0,\ a-2x-y=0 \quad y=a-2x$

$f_y=ax-x^2-2xy=0$

$x(a-x-2y)=0$

$x=0;\ a-x-2y=0$

Sub $y=0$ in f_y

$\therefore x=0;\ x=a$

Sub $y=a-2x$ in f_y

$ax-x^2-2x(a-2x)=0$

$ax-x^2-2ax+4x^2=0$

$3x^2-ax=0$

$x(3x-a)=0$

$x=a/3$

$\therefore$ The possible pairs are $(0,0)\ (0,a)\ (a,0)\ (a/3, a/3)$

$\therefore r=\frac{\partial^2 f}{\partial x^2}=-2y$

$s=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=a-2x-2y$

$t=\frac{\partial^2 f}{\partial y^2}=-2x$

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Pair of values	r	t	rt	s	s^2	$rt-s^2$	result	
$(0,0)$	0	0	0	0	a^2	$-a^2$	$rt-s^2<0$ $r=0$ saddle point	$f(0,0)=0$
$(0,a)$	$-2a$	0	0	$-a$	a^2	$-a^2$	$rt-s^2<0$ $r<0$	$f(0,0)=0$
$(a,0)$	0	$-2a$	0	$-a$	a^2	$-a^2$	$rt-s^2<0$ $r=0$	$f(a,0)=0$
$\left(\frac{a}{3},\frac{a}{3}\right)$	$-\frac{2a}{3}$	$-\frac{2a}{3}$	$\frac{4a^2}{3}$	$\frac{-a}{3}$	$\frac{a^2}{9}$	$\frac{3a^2}{9}=\frac{a^2}{3}$	$rt-s^2>0$ $r<0$ max	$f\left(\frac{a}{3},\frac{a}{3}\right)$ max

$\therefore f(x,y)=xyz$

$$=\left(\frac{a}{3}\right)\left(\frac{a}{3}\right)z$$

$$x+y+z=a$$

$$\frac{a}{3}+\frac{a}{3}+z=a$$

$$z=a-\frac{a}{3}-\frac{a}{3}$$

$$\boxed{z=\frac{a}{3}}$$

The point $\left(\frac{a}{3},\frac{a}{3}\right)$ is a max

$\therefore$ The three positive numbers are

$$x=\frac{a}{3},\ y=\frac{a}{3},\ z=\frac{a}{3}$$

(18) Obtain terms upto the third degree in the Taylor's series expansion of $e^x \sin y$ around the point $(1, \pi/2)$

Sol:-

$$f(x,y)=f(a,b)+\frac{1}{1!}\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]+$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$

$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)+3(x-a)(y-b)^2\right.$$

$$\left.f_{xyy}(a,b)+(y-b)^3 f_{yyy}(a,b)\right]+\dots$$

$f(x,y) = e^x \sin y$ | $f_y(x,y) = e^x \cos y$ | $f_{xy} = e^x \cos y$

$f_x(x,y) = e^x \sin y$ | $f_{yy}(x,y) = -e^x \sin y$ | $f_{xxy} = e^x \cos y$

$f_{xx}(x,y) = e^x \sin y$ | $f_{yyy}(x,y) = -e^x \cos y$ | $f_{xyy} = -e^x \sin y$

$f_{xxx}(x,y) = e^x \sin y$

$(1, \pi/2)$ is the point

$f(x,y) = e^1 \sin(\pi/2) = e$ | $f_y(x,y) = 0$ | $f_{xy} = 0$

$f_x(x,y) = e$ | $f_{yy}(x,y) = -e$ | $f_{xxy} = 0$

$f_{xx}(x,y) = e$ | $f_{yyy}(x,y) = 0$ | $f_{xyy} = -e$

$f_{xxx}(x,y) = e$

$\Rightarrow f(1,\pi/2) + \frac{1}{1!}\left[(x-1)f_x(1,\pi/2) + (y-\pi/2)f_y(1,\pi/2)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,\pi/2)\right.$
$+2(x-1)(y-\pi/2)f_{xy}(1,\pi/2) + \left.(y-\pi/2)^2 f_{yy}(1,\pi/2)\right] + \frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,\pi/2)\right.$
$+3(x-1)^2(y-\pi/2)f_{xxy}(1,\pi/2) + 3(x-1)(y-\pi/2)^2 f_{xyy}(1,\pi/2) + (y-\pi/2)^3 f_{yyy}$
$\left.(1,\pi/2)\right] + \ldots$

$\Rightarrow e + \left[(x+1)(e) + (y-\pi/2)(0)\right] + \frac{1}{2}\left[(x-1)^2(e) + 2(x-1)(y-\pi/2)(0) +\right.$
$\left.(y+\pi/2)^2(-e)\right] + \frac{1}{3!}\left[(x-1)^3(e) + 3(x-1)^2(y-\pi/2)(0) + 3(x+1)(y-\pi/2)^2(-e)\right.$
$\left.+ (y-\pi/2)^3(0)\right] + \ldots$

$\Rightarrow e + \left[(x-1)(e)\right] + \frac{1}{2}\left[(x-1)^2(e) + (y-\pi/2)^2(-e)\right] + \frac{1}{6}(x-1)^3(e) + 3(x-1)(y-\pi/2)^2(-e$

$\Rightarrow e + [ex - e] + \frac{1}{2}\left[(ex^2 + e - 2xe) + (y^2 + \frac{\pi^2}{4} - 2y(\pi/2))(-e)\right] + \frac{1}{6}\left[(x^3 - 3x^2 + 3x - 1)(e\right.$
$\left.+ 0\left[(3x-3)(y^2 + \frac{\pi^2}{4} - 2y(\pi/2)\right\}(-e)\right]$

$\Rightarrow e + [ex - e] + \frac{1}{2}\left[(ex^2 + e - 2xe) - ey^2 - e\left(\frac{\pi^2}{4}\right) + e\pi y\right] + \frac{1}{6}\left[(x^3 e - 3x^2 e + 3xe\right.$
$\left.+ \left\{-3xy^2(e) - 3x\frac{\pi^2}{4}(e) + 3\pi xe + 3y^2(e) + \frac{3\pi^2}{4}e - 3\pi e\right]\right.$ //

(19) Find the Taylor's series expansion of e^{xy} near the point $(1,1)$ upto the second degree terms.

Sol Taylor's series:-

$$f(x,y) = f(a,b) + \frac{1}{1!}[(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots$$

$f(x,y) = e^{xy}$

$f_x(x,y) = e^{xy}(y)$ $\qquad$ $f_y(x,y) = e^{xy}(x)$

$f_{xx}(x,y) = y \cdot e^{xy}(y)$ $\qquad$ $f_{yy}(x,y) = e^{xy}(x)(x) =$

$= y^2 e^{xy}$ $\qquad$ $= x^2 e^{xy}$

$f_{xy}(x,y) = e^{xy}(1) + x \cdot e^{xy} y$

$= e^{xy} + xye^{xy}$

$(1,1)$ is the point

$f_x(1,1) = e$ $\qquad$ $f_y(1,1) = e$

$f_{xx}(1,1) = e$ $\qquad$ $f_{yy}(1,1) = e$

$$f(x,y) = e + \frac{1}{1!}[(x-1)e - (y-1)e] + \frac{1}{2!}[(x-1)^2 e + 2(x-1)(y-1)2e + (y-1)^2 e]$$

$$= e + [ex - e + ye - e] + \frac{1}{2!}[(x^2+1-2x)e + 2(xy-x-y+1)2e + (y^2+1-2y)e]$$

$$= \cancel{e} + ex - \cancel{e} + ye - e + \frac{1}{2}[x^2 e - e - 2xe + (2xy - 2x - 2y + 2)2e + y^2 e - e - $$

$$= e\cancel{x} + y\cancel{e} - e + \frac{1}{2}x^2 e - \frac{1}{2}e - \cancel{x}\cancel{e} + 2xye - 2xe - 2ye + 2e + \frac{1}{2}y^2 e - \frac{1}{2}e - y$$

$$= \frac{ex^2}{2} + \frac{ey^2}{2} - \cancel{2e} + 2xye - 2xe - 2ye + 2\cancel{e}$$

$$f(x,y) = \frac{ex^2}{2} + \frac{ey^2}{2} + 2xye - 2xe - 2ye + \text{———} \; //$$

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Unit II: Multi Variable Calculus

1. If $\phi(x,y,z)=x^2y+y^2x+z^2$, then find $\nabla\phi$ at the point (1,1,1). (K2)
2. Find the directional derivative of $4x^2z+xy^2$ at the point (1,-1,2) in the direction of the vector $2\bar{i}-\bar{j}+3\bar{k}$ (K2)
3. Find the directional derivative of $xyz-xy^2z^3$ at the point (1,2,-1) in the direction of the vector $\bar{i}-\bar{j}-3\bar{k}$ (K2)
4. Find the angle between the normals to the surface $xy-z^2=0$ at the points (1,4,-2) and (3,-3,3) (K2)
5. Find the angle between the normals to the surfaces $xy = z^2$ at the point (1,1,1) and (4,1,2) (K2)
6. Find the angle between the surfaces $z=x^2+y^2-3$ and $x^2+y^2+z^2=9$ at (2,-1,2) (K2)
7. If $\mathrm{F}=\mathrm{x}^3\mathrm{i}+\mathrm{y}^3\mathrm{j}+\mathrm{z}^3\mathrm{k}$, then find curl F (K2)
8. If $\vec{F}=xz^3\vec{i}-2x^2y\vec{j}+2yz^4\bar{k}$, find div F at (1, -1,1). (K2)
9. Prove that div $\bar{r}$ = 3 and curl $\bar{r}$ = 0. (K2)
10. Define Solenoidal. (K1)
11. Find grad φ if φ = xyz at (1, 1, 1). (K2)
12. Show that the vector $2xyi+(x^2+2yz)j+(y^2+1)k$ is irrotational. (K2)
13. Find the constant"a" if the divergence of the vector F = $(x+z)\vec{i}+(3x+ay)\vec{j}+(x-5z)\vec{k}$ is zero (K2)
14. Find the divergence of the vector point function $xy^2\vec{i}+2x^2yz\vec{j}-3yz^2\vec{k}$ (K2)
15. State Green's theorem. (K1)
16. State Stoke's theorem (K1)
17. State Gauss divergence theorem (K1)
18. Use divergence theorem, evaluate $\iint_S \mathrm{r}.\hat{\mathrm{n}}\,\mathrm{ds}$, S is the surface of the sphere $x^2+y^2+z^2=9$

Assignment-3.

Unit-3 Part-A.

1) If $\phi(x,y,z) = x^2y + y^2x + z^2$, then find $\nabla\phi$ at the point $(1,1,1)$

Sol:- $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$.

$\nabla\phi = \frac{\partial}{\partial x}(x^2y + y^2x + z^2)\vec{i} + \frac{\partial}{\partial y}(x^2y + y^2x + z^2) + \frac{\partial}{\partial z}(x^2y + y^2x + z^2)\vec{k}$

$\nabla\phi = (2xy + y^2)\vec{i} + (x^2 + 2yx)\vec{j} + (2z)\vec{k}$

$\nabla\phi_{(1,1,1)} = [2+1]\vec{i} + [1+2]\vec{j} + [2]\vec{k}$.

$= 3\vec{i} + 3\vec{j} + 2\vec{k}$

2) Find the directional derivative of $4x^2z + xy^2$ at the point $(1,-1,2)$ in the directional of the vector $2\vec{i} - \vec{j} + 3\vec{k}$.

Sol $\phi = 4x^2z + xy^2$

$\nabla\phi = \frac{\partial}{\partial x}(4x^2z + xy^2)\vec{i} + \frac{\partial}{\partial y}(4x^2z + xy^2)\vec{j} + \frac{\partial}{\partial \cancel{y}z}(4x^2z + xy^2)\vec{k}$

$\nabla\phi = (8xz + y^2)\vec{i} + (2xy)\vec{j} + (4x)\vec{k}$

$\nabla\phi$ ₹ $(1,-1,2) = [16+1]\vec{i} + [-2]\vec{j} + 4\vec{k}$

$= 17\vec{i} - 2\vec{j} + 4\vec{k}$

Directional derivate $= \frac{\nabla\phi}{|\nabla\phi|} = \frac{17\vec{i} - 2\vec{j} + 4\vec{k}}{\sqrt{14}}$

3. Find D.D of $xyz - xy^2z^3$ at the point $(1,2,-1)$ in the direction of vector $\vec{i} - \vec{j} - 3\vec{k}$

Sol: $D.D = \frac{\nabla\phi}{|\nabla\phi|}$

$\phi = xyz - xy^2z^3$

$\nabla\phi = \frac{\partial}{\partial x}(xyz - xy^2z^3) + \frac{\partial}{\partial y}(xyz - xy^2z^3)\vec{j} + \frac{\partial}{\partial z}(xyz - xy^2z^3)\vec{k}$

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$\nabla\phi_{(1,2,-1)} = [yz - y^2z^3]\vec{i} + [xz - 2xyz^3]\vec{j} + [xy - 3xy^2z^3]\vec{k}$

$\nabla\phi_{(1,2,-1)} = [-2+4]\vec{i} + [-1+4]\vec{j} + [2-12]\vec{k}$

$= 2\vec{i} + 3\vec{j} - 10\vec{k}$

$$D.D = \frac{2\vec{i} + 3\vec{j} - 10\vec{k}}{\sqrt{(-1)^2+(1)^2+(-3)^2}} = \frac{2\vec{i} + 3\vec{j} - 10\vec{k}}{\sqrt{11}}.$$

4. Find the angle between the normal to the surface $xy - z^2 = 0$ at the point $(1, 4, -2)$ and $(3, -3, 3)$

Sol

$\phi = xy - z^2$

$\nabla\phi = \frac{\partial}{\partial x}(xy - z^2)\vec{i} + \frac{\partial}{\partial y}(xy - z^2)\vec{j} + \frac{\partial}{\partial z}(xy - z^2)\vec{k}$

$= y\vec{i} + x\vec{j} + (-2z)\vec{k}$

$\nabla\phi_{(1,4,2)} = 4\vec{i} + \vec{j} + 4\vec{k}$

$|\nabla\phi_1| = \sqrt{16+1+16} = \sqrt{33}$

$\nabla\phi_2(3,-3,3) = -3\vec{i} + 3\vec{j} - 6\vec{k}$

$|\nabla\phi_2|_{(3,-3,3)} = -3\vec{i} + 3\vec{j} + 6\vec{k}$

$= \sqrt{9+9+36}$

$|\nabla\phi_2|_{(3,-3,3)} = \sqrt{54}.$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|} = \frac{(4\vec{i} + \vec{j} + 4\vec{k})(-3\vec{i} + 3\vec{j} - 6\vec{k})}{\sqrt{33}\sqrt{54}.}$$

$$= \frac{-12 + 3 - 24}{\sqrt{33}\sqrt{54}} = \frac{-33}{\sqrt{33}\sqrt{54}}.$$

5. Find the angle between the normals to surface $xy - z^2 = 0$ at the point $(1,1,1)$ & $(4,1,2)$.

Sol:- $\phi = xy - z^2$

$$\nabla\phi = \frac{\partial}{\partial x}(xy - z^2)\vec{i} + \frac{\partial}{\partial y}(xy - z^2)\vec{j} + \frac{\partial}{\partial z}(xy - z^2)$$

$$\nabla\phi = [y]\vec{i} + [x]\vec{j} + [-2z]\vec{k}$$

$$\nabla\phi_{(1,1,1)} = \vec{i} + \vec{j} - 2\vec{k}$$

$$|\nabla\phi_1| = \sqrt{1+1+4} = \sqrt{6}$$

$$\nabla\phi_2(4,1,2) = \vec{i} + 4\vec{j} - 4\vec{k}$$

$$|\nabla\phi_2|_{(4,1,2)} = \sqrt{1+16+16} = \sqrt{33}$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{(\vec{i} + \vec{j} + 2\vec{k})(\vec{i} + 4\vec{j} + 4\vec{k})}{\sqrt{6}\sqrt{33}}$$

$$= \frac{1+4+8}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{6}\sqrt{33}}$$

6. Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 - 9 = 0$ at $(2,-1,2)$

$$\phi_1 = x^2 + y^2 - z - 3 = 0$$

$$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi_1(2,-1,2) = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\phi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$\nabla\phi_2 = x^2 + y^2 + z^2 - 9 = 0$$

$$\nabla\phi_2 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}.$$

$$\nabla\phi_2(2,-1,2) = 4\vec{i} + 2\vec{j} + 4\vec{k} = \sqrt{16+4+16}$$

$$= \sqrt{36} = 6$$

$$\cos\theta = \frac{(4\vec{i}+2\vec{j}-\vec{k})(4\vec{i}-2\vec{j}+4\vec{k})}{\sqrt{21}\cdot 6}$$

$$= \frac{16+4-4}{6\sqrt{21}}$$

$$\cos\theta = \frac{16}{6\sqrt{21}}$$

7. If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ then find Curl F

Sol Curl $\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z^3) - \frac{\partial}{\partial z}(y^3)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z^3) - \frac{\partial}{\partial z}(x^3)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial y}(x^3)\right]$$

Curl $\vec{F} = (0)\vec{i} - (0)\vec{j} + (0)\vec{k}$

8. If $\vec{F} = xz^3\vec{i} - 2x^2y^2\vec{j} + 2yz^4\vec{k}$, find div F at $(1,-1,1)$

Sol: div $\vec{F} = \frac{\partial}{\partial x}(xz^3)\vec{i}\cdot\vec{i} - 2\frac{\partial}{\partial y}(x^2y)\vec{j}\cdot\vec{j} + 2\frac{\partial}{\partial z}(yz^4)\vec{k}\cdot\vec{k}$

div $\vec{F}$. $\nabla\cdot\vec{F} = z^3 - 2x^2 + 8yz^3$.

$\nabla\cdot\vec{F}$ at $(1,-1,1) = 1-2-8$
$= -9$.

9. Prove that div $\vec{r} = 3$ and Curl $\vec{r} = 0$

Sol: $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\nabla\cdot\vec{r} = \frac{\partial}{\partial x}(x)\vec{i}\cdot\vec{i} + \frac{\partial}{\partial y}(y)\vec{j}\cdot\vec{j} + \frac{\partial}{\partial z}(z)\vec{k}\cdot\vec{k}$$

div $\vec{r} = \nabla\cdot\vec{r} = 1+1+1 = 3$

$$\text{Curl } \vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$\nabla \times \vec{r} = 0$$

Vector Calculus

1 If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$; find ϕ if $\varphi(-1,2,2) = 4$. (K4)

2. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ prove that i) $\nabla r = \frac{\vec{r}}{r}$ ii) $\nabla r^n = nr^{n-2}\vec{r}$. where $r = |\vec{r}|$ (K4)

3.Show that the vector $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$ is both solenoidal and irrotational. (K5)

4 Show that the vector $2xy\vec{i} + (x^2+2yz)\vec{j} + (y^2+1)\vec{k}$ is irrotational. (K4)

5. Find the value of the constant a, b, c so that the vector

$\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational. (K4)

6. Evaluate $\nabla.(r^3\vec{r})$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ (K4)

7.If $\vec{F} = 3xyz^2\vec{i} + 2xy^3\vec{j} - x^2yz\vec{k}$ & $f = 3x^2 - yz$ find i) $\nabla.\nabla f$ .ii) div curl $\vec{F}$ at (1,-1, 1) (K4)

8. If $\phi = 3x^2z - y^2z^3 + 4x^3y - 2x - 3y - 5$ find $\nabla^2\phi$ (K4)

9. If $\vec{F} = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$, find curl curl $\vec{F}$. (K4)

10. If $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla^2 r^n = n(n+1)r^{n-2}$ and hence deduce $\nabla\left(\frac{1}{r}\right)$.

11. Show that $\nabla^2(r^n r) = n(n+3)r^{n-2}\, r$ with usual notations. (K4)

12. Find div(grad φ) and, curl (gradϕ) at (1,1,1) for $\phi = x^2y^3z^4$ (K4)

13. For $\varphi = x^3 + y^3 + 3xyz; \vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ find i) curl grad ϕ ii) div curl $\vec{F}$

14. Find a and b such that the surfaces $ax^2 - byz = (a+2x)$ and $4x^2y + z^2 = 4$ cut orthogonally at (1,1,1). (K5)

15. Find a and b such that the surfaces $ax^3 - by2z = (a+3)x^2$ and $4x^2y + z^2 = 11$ cut orthogonally at (1,1,1). (K5)

16. Verify Green's theorem in the XY plane for $\int_C (xy + y^2)dx + x^2dy$,

where C is the closed curve of the region bounded by $y = x$ and $y = x^2$ (K4)

17. Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region bounded by $x = 0, y = 0, x + y = 1$ (K4)

18. Verify Green`s theorem for $\int_c \left[\left(x^2 - y^2\right)dx + 2xydy\right]$,where C is the boundary of the rectangle in the XOY-plane bounded by the lines x = 0, x = a, y = 0 and y = b. (K4)

19. By using Green's theorem, Evaluate $\int_c \left\{\left(2x^2 - y^2\right)dx + \left(x^2 + y^2\right)dy\right\}$,where C is the boundary in the XY plane of the area enclosed by the X axis and the semi-circle $x^2 + y^2 = 1$ in the upper half XY plane. (K5)

20. Verify Green`s theorem in a plane for the integral $\int_C ((x - 2y)dx + xdy)$, taken around the circle $x^2 + y^2 = 1$ (K5)

21.Verify Gauss divergence theorem for $\overline{F} = (x^2 - yz)\overline{i} + (y^2 - zx)\overline{j} + (z^2 - xy)\overline{k}$ taken over the rectangular parallelepiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$ (K5)

22. Verify Gauss-divergence theorem for the vector function $\overrightarrow{F} = 4xz\overrightarrow{i} - y^2\overrightarrow{j} + yzk$ over the Cube bounded by $x = 0, y = 0, z = 0$ and $x = 1, y = 1, z = 1$ (K5)

23. Verify Gauss-divergence theorem for $\overrightarrow{F} = 4x\overrightarrow{i} - 2y^2\overrightarrow{j} + z^2\overline{k}$ taken over the region by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$ (K5)

24. Using Gauss divergence theorem, evaluate $\iint_S F \cdot \hat{n}\, ds$ where over the surface $F = x^3 i + y^3 j + z^3 k$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ (K4)

25. Verify Stoke's theorem for the function $z = 0$ integrated round the square in the z = 0 plane whose sides are along the lines x = 0, y = 0, x = a, y = a. (K5)

26. Verify Stoke's theorem for a vector field $\overline{F} = (x^2 - y^2)\overline{i} + 2xy\overline{k}$ in the rectangular

region of XOY-plane bounded by the lines $x = -a, x = a, y = 0, y = b$. (K5)

27.Verify Stoke's theorem for a vector field $\overline{F} = (y - z + 2)\,\overline{i} + (yz + 4)\,\overline{j} - xz\,\overline{k}$ where S is the surface of the cube $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$ above the XY-plane. (K5)

28. Verify stokes theorem for $\overrightarrow{F} = y^2 z\overrightarrow{i} + z^2 x\overrightarrow{j} + x^2 y\overline{k}$ where S is the open surface of the cube formed by the planes x = - a, x = a, y = - a, y = a, z = -a, z = a in which z =-a is cut open. (K5)

Assignment - 4
Unit - 2

Part - B.

1. If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$. Find ϕ if $\phi(-1,2,2)$ = 4.

A $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^3\vec{k}$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$$

equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$

$$\vec{i} = \frac{\partial\phi}{\partial x} = 2xyz^3$$

$$\phi = \int 2xyz^3 dx = \frac{2x^2}{2}yz^3 + f(y,z)$$

$$= x^2yz^3 + f(y,z) \quad \text{—} (1)$$

$$\vec{j} = \frac{\partial\phi}{\partial y} = x^2z^3$$

$$\phi = \int x^2z^3 dy = x^2yz^3 + f(x,z) \quad \text{—} (2)$$

$$\vec{k} = \frac{\partial\phi}{\partial z} = 3x^2yz^3$$

$$\phi = \int 3x^2yz^3 dz = x^2yz^3 + f(x,y) \quad \text{—} (3)$$

from (1), (2) and (3)

$$\phi = x^2yz^3 + c$$

G.T $\phi(-1,2,2) = 4$

$$\Rightarrow \phi = (-1)^2(2)(2^3) + c = 4$$

$$16 + c = 4$$

$$c = -12$$

$$\therefore \phi = x^2yz^3 - 12$$

2 If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ prove that

(i) $\nabla r = \frac{\vec{r}}{r}$ (ii) $\nabla r^n = nr^{n-2}\vec{r}$ where $r = |\vec{r}|$

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Sol:- (i) Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r = |\vec{r}| = (x^2 + y^2 + z^2)^{1/2}$$

let $f(x,y,z) = r = (x^2+y^2+z^2)^{1/2}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x) = \frac{x}{r} \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y) = \frac{y}{r} \quad \text{--- (2)}$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z) = \frac{z}{r} \quad \text{--- (3)}$$

$$\nabla r = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\nabla r = \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}.$$

$$\nabla r = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}$$

$$\nabla r = \frac{\vec{r}}{r}$$

hence proved.

(ii) $r = (x^2+y^2+z^2)^{1/2}$

$$r^n = (x^2+y^2+z^2)^{n/2}.$$

let $f(x,y,z) = r^n = (x^2+y^2+z^2)^{n/2}$

$$\frac{\partial f}{\partial x} = \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x)$$

$$\frac{n}{2}(r)^{n-2}(2x) = n.x.r^{n-2}$$

lly, $\frac{\partial f}{\partial y} = ny.r^{n-2}$; $\frac{\partial f}{\partial z} = n.z.r^{n-2}$

$$\nabla r^n = \left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right]f$$

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$= n.x\, r^{n-2}\vec{i} + n.y\, r^{n-2}\vec{j} + nz\, r^{n-2}\vec{k}$$

$$= n.r^{n-2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= n.\, r^{n-2}\, \vec{r}$$

$$\nabla r^n = n.\, r^{n-2}\, (\vec{r})$$

hence proved.

3. Show that the vector $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$ is both solenoidal and irrotational.

Sol:- $\nabla\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\vec{F}$

$$= \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x)\vec{i} + \frac{\partial}{\partial y}(3xz + 2xy)\vec{j} + \frac{\partial}{\partial z}(3xy - 2xz + 2z)\vec{k}$$

$$= (-2)\vec{i} + 2x\vec{j} + (2 - 2x)\vec{k}$$

$$= 0.$$

$\nabla.\vec{F} = 0$; $\vec{F}$ is solenoidal.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z) \end{vmatrix}$$

$$\nabla\times\vec{F} = \vec{i}\left[\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xy + 2xy)\right] + \vec{j}\left[\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x)\right] + \vec{k}\left[\frac{\partial}{\partial x}(3xy + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right]$$

$$\nabla\times\vec{F} = \vec{i}(0) + \vec{j}(0) + \vec{k}(0) = 0$$

$\therefore \vec{F}$ is both solenoidal and irrotational.

4. Show that the vector $2xy\vec{i} + (x^2+2yz)\vec{j} + (y^2+1)\vec{k}$ is irrotational

Sol: let $\vec{F} = 2xy\vec{i} + (x^2+2yz)\vec{j} + (y^2+1)\vec{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+2yz & y^2+1 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(y^2+1) - \frac{\partial}{\partial z}(x^2+2yz)\right) - \vec{j}\left(\frac{\partial}{\partial x}(y^2+1) - \frac{\partial}{\partial z}(2x)\right) + \vec{k}\left(\frac{\partial}{\partial x}(x^2+2yz) - \frac{\partial}{\partial y}(2xy)\right)$$

$$= \vec{i}(2y-2y) - \vec{j}(0-0) + \vec{k}(2x-2x)$$

$$= 0.$$

$\therefore \nabla\times\vec{F} = 0.$

$\vec{F}$ is a irrotational vector.

5. Find the value of the constant a, b, c so that the vector $\vec{F} = (x+2y+az)\vec{i} + (bx-3y+z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Sol: If $\vec{F}$ is irrotational.

$\nabla\times\vec{F} = 0$

$$\begin{vmatrix} i & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0.$$

$$\vec{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right] + \vec{k}\left[\frac{\partial}{\partial x}(bx-3y+z) - \frac{\partial}{\partial y}(x+2y+az)\right] = 0$$

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$\nabla \times \vec{F} = \vec{i}\left[\frac{\partial}{\partial y}(y^2+1) - \frac{\partial}{\partial z}(x^2+2yz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(y^2+1) - \frac{\partial}{\partial z}(2x)\right] + \vec{k}\left[\frac{\partial}{\partial x}(x^2+2yz) - \frac{\partial}{\partial y}(2xy)\right]$

$= \vec{i}\,(2y-2y) - \vec{j}\,(0-0) + \vec{k}\,(2x-2x)$

$= 0$

$\nabla \times \vec{F} = 0$

$\vec{F}$ is a irrotational vector.

$\vec{i}\,(c-1) - \vec{j}\,(4-a) + \vec{k}\,(b-2) = 0$

$c-1=0$; $4=a$; $b-2=0$

$c=1$ $\qquad\qquad\qquad\qquad$ $b=2$

The values of constants are

$a=4$, $b=2$, $c=1$

6. Evaluate $\nabla \cdot (r^3 \vec{r})$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r = \sqrt{x^2+y^2+z^2}$

$r^3 = (x^2+y^2+z^2)^{3/2}$

$\nabla \cdot (r^3 \vec{r}) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (x^2+y^2+z^2)^{3/2}(x\vec{i} + y\vec{j} + z\vec{k})$ -

$\frac{\partial}{\partial x}\left(x(x^2+y^2+z^2)^{3/2}\right) + \frac{\partial}{\partial y}\left(y(x^2+y^2+z^2)^{3/2}\right) + \frac{\partial}{\partial z}\left(z(x^2+y^2+z^2)^{3/2}\right)$ —— (1)

$\frac{\partial}{\partial x}\left(x(x^2+y^2+z^2)^{3/2}\right) = (x^2+y^2+z^2)^{3/2} + x \cdot \frac{3}{2}(x^2+y^2+z^2)^{\frac{3}{2}-1}(2x)$

$= r^3 + x^2(3)(r)$ —— (2)

similarly

$$\frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{3/2}\right] = (x^2+y^2+z^2)^{3/2} + y\cdot\frac{3}{2}(x^2+y^2+z^2)^{3/2-1}\cdot(2y)$$

$$= r^3+y^2(3)r \quad \text{--- (3)}$$

$$\frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{3/2}\right] = r^3+3z^2r \quad \text{--- (4)}$$

Sub (2), (3), (4) in eqn (1)

$$\nabla(r^3\vec{r}) = \left[r^3+3x^2r\right]+\left[r^3+3y^2r\right]+\left[r^3+3z^2r\right].$$

$$= 3r^3+3r(x^2+y^2+z^2)$$

$$= 3r^3+3r(r^2)$$

$$= 6r^3$$

$$\nabla(r^3.\vec{r}) = 6r^3$$

Hence proved.

7. If $\vec{F} = 3xyz^2\vec{i}+2xy^3\vec{j}-x^2yz\vec{k}$ and $f = 3x^2-yz$, find (i) $\nabla.\nabla f$ (ii) div curl $\vec{F}$ at $(1,-1,1)$.

Sol (i) $\nabla f = \left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)(3x^2-yz)$

$$\nabla f = 6x\vec{i}+y\vec{k}-z\vec{j}$$

$$\nabla.\nabla f = \left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)(6x\vec{i}-z\vec{j}-y\vec{k})$$

$$= \left[\frac{\partial}{\partial x}(6x)+\frac{\partial}{\partial y}(-z)+\frac{\partial}{\partial z}(-y)\right]$$

$$\nabla.\nabla f = 6.$$

(ii) Curl $\vec{F} = \nabla\times\vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3xyz^2 & 8xy^3 & -x^2yz \end{vmatrix}$$

$= \vec{i}\,(-x^2z - 0) - j\,(-2xyz - 6xyz) + \vec{k}\,(2y^3 - 3xz^2)$

$= -x^2z\vec{i} + 8xyz\vec{j} + \vec{k}\,(2y^3 - 3xz^2)$

$\text{div Curl } \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(-x^2z\vec{i} + 8xyz\vec{j} + k\,(2y^3 - 3xz^2)\right)$

$\text{div Curl } \vec{F} = -2xz + 8xz - 6xz$

$\text{div Curl } \vec{F} = 0$

8) If $\phi = 3x^2z - y^2z^3 + 4x^3y - 2x - 3y - 5$. Find $\nabla^2\phi$.

Sol: $\nabla\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(\phi)$

$= \left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right](3x^2z - y^2z^3 + 4x^3y - 2x - 3y - 5)$

$= (6xz + 12x^2y - 2)\,\vec{i} + (4x^3 - 2yz^3 - 3)\,\vec{j} + (3x^2 - 3y^2z^2)\,\vec{k}$

$\nabla\cdot(\nabla\phi) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left[(6xz + 12x^2y - 2)\vec{i} + (4x^3 - 2yz^3 - 3)\vec{j} + (3x^2 - 3y^2z^2)\vec{k}\right].$

$= \frac{\partial}{\partial x}(6xz + 12x^2y - 2) + \frac{\partial}{\partial y}(4x^2 - 2yz^3 - 3) + \frac{\partial}{\partial z}(3x^2 - 3y^2z^2)\,\vec{k}.$

$= 6z + 24xy - 2z^3 - 6y^2z$

$\nabla^2\phi = 6z + 24xy - 2z^3 - 6y^2z$

9. If $\vec{F} = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$. find Curl. Curl $\vec{F}$

Sol $\text{Curl } \vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2x \end{vmatrix}$$

$= \vec{i}\left[\frac{\partial}{\partial y}(z^2x) - \frac{\partial}{\partial z}(y^2z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z^2x) - \frac{\partial}{\partial z}(x^2y)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y^2z) - \frac{\partial}{\partial y}(x^2y)\right]$

$= \vec{i}\,(-y^2) - \vec{j}\,(z^2) + \vec{k}\,(-x^2)$

$= -y^2\vec{i} - z^2\vec{j} - x^2\vec{k}$

$$\text{Curl Curl } \vec{F} = \nabla \times \text{Curl } \vec{F} = \nabla \times (\nabla \times \vec{F})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & -z^2 & -x^2 \end{vmatrix}$$

$$= \vec{i}\,(2z) - \vec{j}\,(-2x) + \vec{k}\,(2y)$$

$$= 2z\vec{i} + 2x\vec{j} + 2y\vec{k}$$

$$\therefore \text{Curl Curl } \vec{F} = 2z\vec{i} + 2x\vec{j} + 2y\vec{k}$$

10. If $\vec{r}$ is the position vector of the point (x, y, z). prove that $\nabla^2 r^n = n(n+1)\, r^{n-2}$ and hence deduce $\nabla\left(\frac{1}{r}\right)$

sol:- $\nabla^2 r^n = \nabla \cdot (\nabla \cdot r^n)$ —— ①

$$\nabla r^n = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(r^n) \text{ —— ②}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

now, $\frac{\partial r^n}{\partial x} = \frac{n}{2}(2x) \cdot (x^2+y^2+z^2)^{n/2-1} = nx \cdot r^{n-2}$ —— ③

lly $\frac{\partial r^n}{\partial x} = ny\, r^{n-2}$ —— (4) ; $\frac{\partial r^n}{\partial z} = nz\, r^{n-2}$ —— ⑤

Sub 3, 4, 5 in eqn ②

$$\nabla r^n = n r^{n-2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla r^n = n \cdot r^{n-2}(\vec{r}) \text{ —— ⑥}$$

from ① & ⑥

$$\nabla^2(r^n) = \nabla(\nabla r^n) = \nabla(n \cdot r^{n-2} \cdot \vec{r})$$

$$= n\left[(\nabla r^{n-2})\vec{r} + r^{n-2}(\nabla\vec{r})\right]$$

$= n[(n-2)r^{n-4}.\vec{r}.\vec{r} + r^{n-2}(3)]$

$\nabla^2(r^n) = n(n+1)r^{n-2}$

$\nabla\left(\frac{1}{r}\right) = \nabla(r^{-1})$

Put $n=-1$ in eqn ⑥

$\nabla(r^{-1}) = (-1)r^{-1-2}\vec{r} = \frac{-\vec{r}}{r^3}$

$\nabla^2(r^{-1}) = (-1)(-1+1)r^{-1-2} = 0$

11. Show that $\nabla^2(r^n\vec{r}) = n(n+3)r^{n-2}\vec{r}$ with usual notations

Soln: $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r = \sqrt{x^2+y^2+z^2}$

$r^n = (x^2+y^2+z^2)^{n/2}$

$\nabla.(r^n.\vec{r}) = \nabla(x^2+y^2+z^2)^{n/2}.(x\vec{i} + y\vec{j} + z\vec{k})$

$\frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{n/2}\right] = (x^2+y^2+z^2)^{n/2} + x.\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x)$

$= r^n + n.x^2.r^{n-2}$ ——①

Similarly $\frac{\partial}{\partial y}\left[y(y^2+x^2+z^2)^{n/2}\right] = r^n + n.y^2.r^{n-2}$ ——②

$\frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{n/2}\right] = r^n + n.z^2r^{n-2}$ ——③

from ①, ② and ③

$\nabla(r^n.\vec{r}) = 3r^n + n.r^{n-2}(x^2+y^2+z^2)$

$= 3r^n + nr^2r^{n-2}$

$= 3r^n + n.r^n$

$= (n+3)r^n$ ——④

$\nabla^2(r^n.\vec{r}) = \nabla(\nabla r^n\vec{r})$

$= \nabla((n+3)r^n)$

$= (n+3)(\nabla.r^n)$ ——⑤

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$$\nabla r^n = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (x^2+y^2+z^2)^{n/2}$$

$$\frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{n/2}\right] = n/2\,(2x)(x^2+y^2+z^2)^{n/2-1}$$

$$= n.x.r^{n-2}$$

lly $\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} = n.y.r^{n-2}$

$$\nabla r^n = n.r^{n-2}\vec{r}$$

eqn (5) $\Rightarrow$ $\nabla^2(r^n\vec{r}) = (n+3)(nr^{n-2}\vec{r})$

$$\nabla^2(r^n\vec{r}) = n(n+3)\,r^{n-2}\vec{r}$$

hence proved.

Q Find div (grad ϕ) and curl (grad ϕ) at (1,1,1) for $\phi = x^2y^3z^4$

Sol: grad $\phi = \left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right](x^2y^3z^4)$

$$= \frac{\partial}{\partial x}(x^2y^3z^4)\vec{i} + \frac{\partial}{\partial y}(x^2y^3z^4)\vec{j} + \frac{\partial}{\partial z}(x^2y^3z^4)\vec{k}$$

$$= 2xy^3z^4\vec{i} + 3x^2y^2z^4\vec{j} + 4x^2y^3z^3\vec{k}$$

div (grad ϕ) $= \nabla.$ (grad ϕ)

$$= \left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right]\left[2xy^3z^4\vec{i} + 3x^2y^2z^4\vec{j} + 4x^2y^3z^3\vec{k}\right]$$

$$= \frac{\partial}{\partial x}(2xy^3z^4) + \frac{\partial}{\partial y}(3x^2y^2z^4) + \frac{\partial}{\partial z}(4x^2y^3z^3)$$

$$= 2y^3z^4 + x^2yz^4 + 12x^2y^3z^2$$

div (grad ϕ) at (1,1,1) $= 2+6+12$
$= 20$

$$\text{curl (grad } \phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix}$$

$= \vec{i}\,[12x^2y^2z^3 - 12x^2y^2z^3] - \vec{j}\,[8xy^3z^3 - 8xy^3z^3] + \vec{k}\,[6xy^2z^4 - 6xy^2z^4]$

Curl (grad ϕ) at $(1,1,1) = 0$

13. For $\phi = x^3 + y^3 + 3xyz$; $\vec{F} = x^2\vec{i} + y^3\vec{j} + z^3\vec{k}$ find (i) Curl grad ϕ
(ii) div Curl $\vec{F}$

Sol (i) grad $\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(x^3 + y^3 + 3xyz)$

$= (3x^2 + 3yz)\vec{i} + (3y^2 + 3xz)\vec{j} + 3xy\,\vec{k}$

$$\text{Curl (grad } \phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2+3yz & 3y^2+3xz & 3xy \end{vmatrix}$$

$= \vec{i}\,(3x - 3x) - \vec{j}\,(3y - 3y) + \vec{k}\,(3z - 3z)$

$= 0.$

(ii) $$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$= \vec{i}\left[\frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial y}(x^3)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z^3) - \frac{\partial}{\partial z}(x^3)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial y}(x^3)\right]$

$= (0)\vec{i} + (0)\vec{j} + (0)\vec{k}$

$= 0$

div (Curl $\vec{F}$) $= 0$

14. Find a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^2 = 4$ Cut orthogonally at $(1,1,1)$

Sol let $f_1 = ax^2 - byz - a - 2x = 0$ —— (1)

$f_2 = 4x^2y + z^2 - 4 = 0$ —— (2)

f_1 and f_2 cut orthogonally, so

$\nabla f_1 \cdot \nabla f_2 = 0$

$\nabla f_1 = \left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right]\left[ax^2 - byz - a - 2x\right]$

$= (2ax - 2)\vec{i} + (-bz)\vec{j} + (-by)\vec{k}$

$\nabla f_2 = \left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right]\left[4x^2y + z^2 - 4\right]$

$= 8xy\vec{i} + 4x^2\vec{j} + 2z\vec{k}$

$\nabla f_1 \cdot \nabla f_2 = 8xy(2ax - 2) - 4x^2(bz) - 2z(by) = 0$

at $(1,1,1) \Rightarrow 8(2a - 2) - 4b - 2b = 0$

$16a - 16 - 6b = 0$

$2(8a - 3b - 8) = 0$

$\Rightarrow 8a - 3b - 8 = 0 \longrightarrow (3)$

f_1 at $(1,1,1) \Rightarrow a - b - a - 2 = 0 \Rightarrow b = 2$

sub in (3) $\Rightarrow 3a - 3(-2) - 8 = 0$

$8a - 2 = 0$

$a = 1/4, \quad b = -2$

15. Find a and b such that the surfaces $ax^3 - byz = (a+3)x^2$ and $4x^2y + z^2 = 11$ cut orthogonal at $(1,1,1)$

let $f_1 = ax^3 - byz - (a+3)x^2 = 0 \rightarrow (1)$

$f_2 = 4x^2 + z^2 - 11 = 0 \longrightarrow (2)$.

$\nabla f_1 = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(ax^3 - by(2z) - (a+3)x^2\right)$

$= \left[3ax^2 - 2(a+3)x\right]\vec{i} - 2bz\vec{j} - 2by\vec{k}$

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$$\nabla f_2 = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(4x^2y + z^2 - 11)$$

$$= 8xy\,\vec{i} + 4x^2\,\vec{j} + 2z\,\vec{k}$$

f_1 and f_2 cut orthogonally, so $\nabla f_1 \cdot \nabla f_2 = 0$

$\Rightarrow 8xy\,(3ax^2 - 2(a+3)x) - 4x^2(2bz) - 2by(2z) = 0$

at $(1,1,1)$ $\nabla f_1 \cdot \nabla f_2 \Rightarrow$ ~~8(3a)1-~~

$$8(3a - 2a - 6) - 8b - 4b = 0$$

$$\Rightarrow 2a - b - 12 = 0 \quad \text{--- (3)}$$

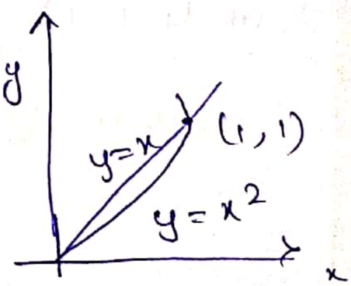
Sub $b = -3/2$ in (3)

$$2a + \frac{3}{2} - 12 = 0$$

$$a = \frac{21}{4}; \quad b = -3/2$$

16 verify Green's theorem in xy plane for $\int_C (xy + y^2)\,dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$

Sol:- $$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$



$P = xy + y^2 \qquad Q = x^2$

$$\frac{\partial P}{\partial y} = x + 2y; \quad \frac{\partial Q}{\partial x} = 2x$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \int_0^1 \int_{x^2}^{x} (x - 2y)\,dy\,dx$$

$$= \int_0^1 (xy - y^2)_{x^2}^{x}\,dx$$

$$= \int_0^1 -(x^3 - x^4)\,dx$$

$$= \left[-\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 \Rightarrow \frac{-1}{4} + \frac{1}{5}$$

$\int_C (P\,dx + Q\,dy) = \int_{C_1} + \int_{C_2}$

along C_1, $y = x^2$ and various from 0 to 1

$$\int_{C_1} = \int_0^1 [x(x^2) + (x^2)^2]\,dx + x^2\,dx^2.$$

$$\int_{C_1} = \int_0^1 (x^3 + x^4)\,dx + x^2(2x)\,dx$$

$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \left(\frac{3}{4} + \frac{1}{5}\right) = \frac{19}{20}$$

along C_2, $y = x$

x varies from 1 to 0.

$$\int_{C_2} = \int_1^0 (x(x) + x^2)\,dx + x^2\,dx$$

$$= \int_1^0 3x^2\,dx = \left(\frac{3x^3}{3}\right)_1^0$$

$$= 0 - 1 = -1$$

$$\int_C P\,dx + Q\,dy = \int_{C_1} + \int_{C_2}$$

$$= \frac{19}{20} - 1 = \frac{-1}{20}$$

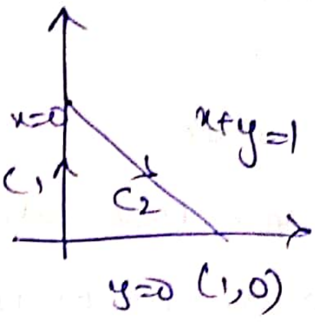
$\therefore \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$\Rightarrow \int_C (xy + y^2)\,dx + x^2\,dy = \frac{-1}{20}$

There fore green's theorem is verified

17. verify green's theorem in plane for $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the boundary of the region bounded by $x=0$, $y=0$.

Sol: $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$



By green's theorem.

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$P = 3x^2 - 8y^2$; $Q = 4y - 6xy$

$\frac{\partial P}{\partial y} = -16y$; $\frac{\partial Q}{\partial x} = -6y$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_0^1\int_0^{1-y} (-6y+16y)\,dx\,dy$$

$$= \int_0^1 (10xy)_0^{1-y}\,dy$$

$$= \int_0^1 [10(1-y)y]\,dy$$

$$= 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= 10\left[\frac{1}{6}\right] = 5/3 \quad \text{--- (1)}$$

$$\int_C P\,dx + Q\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3}.$$

along C_1; $x=0$ y ranges from 0 to 1

$$\int_{C_1} = \int_0 (3x^2 - 8y^2)\,dx + [4(1-x) - 6xy](-dx)$$

$x=0$, $dx=0$ $\Rightarrow$ $\int_{C_1} = \int_0^1 4y\,dy = \left[\frac{4y^2}{2}\right]_0^1 = 2$

along C_2, $y = 1-x$; x ranges from 0 to 1

$$\int_{C_2} = \int_0 (3x^2 - 8(1-x^2))\,dx + (4(1-x) - 6x(1-x))(-dx)$$

$$= \left[\frac{3x^3}{3} - \frac{8(1-x)^3}{(-3)} - \frac{4(1-x)^2}{(-2)} + \frac{6x^2}{2} - \frac{6x^3}{3}\right]_0^1$$

$$= \left[\frac{-11}{3} + \frac{26}{2} - 12\right] = \frac{-8}{3}$$

along C_3 $y=0$; x ranges from 1 to 0

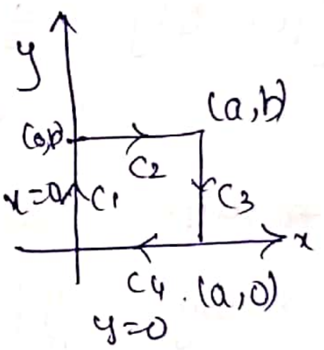
$$\int_{C_3} = \int_1^0 3x^2 dx = \left(\frac{3x^3}{3}\right)_1^0 = -1$$

$$\int_C pdx + qdy = \left|\int_{C_1} + \int_{C_2} + \int_{C_3}\right| = \left|2 - \frac{8}{3} - 1\right| = 5/3 \quad -②$$

from ① & ② green's theorem is verified

18. verify green's theorem for $\int[(x^2-y^2)dx + 2xy\,dy]$ where C is the boundary of the rectangle in xoy plane bounded by the lines $x=0$; $x=a$; $y=0$ and $y=b$

sol. $\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$



$$\frac{\partial P}{\partial y} = -2y ; \quad \frac{\partial Q}{\partial x} = 2y$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_0^a \int_0^b 4y\,dy\,dx$$

$$= \int_0^a \left(\frac{4y^2}{2}\right)_0^b dx$$

$$= \int_0^a (2b^2)\,dx$$

$$= 2b^2(a-0) = 2ab^2 \quad -①$$

$$\int_C pdx + qdy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}.$$

along C_1 ; $x=0$ y ranges from 0 to b.

$$\int_{C_1} = \int_0^b 2xy\,dy = 0 \quad -②$$

along $C_2 \Rightarrow y=b$; x varies from 0 to a.

$$\int_{C_2} = \int_0^a (x^2-b^2)\,dx = \frac{a^3}{3} - ab^2 \quad \text{--- (3)}$$

along $C_3 \Rightarrow x=a$; y is b to 0

$dx=0$

$$\int_{C_3} = \int_b^0 2ay\,dy = \left(\frac{2ay^2}{2}\right)_b^0$$

$$= -ab^2 \quad \text{--- (4)}$$

along $C_4 \Rightarrow y=0$ x varies from a to 0

$$\int_{C_4} = \int_a^0 x^2\,dx = \left[\frac{x^3}{3}\right]_a^0 = \frac{-a^3}{a} \quad \text{--- (5)}$$

$\int_C P\,dx + Q\,dy$ = add (2), (3), (4) & (5)

$$\int_C P\,dx + Q\,dy = \frac{a^3}{3} - ab^2 - ab^2 - \frac{a^3}{3}$$

$$= -2ab^2 \quad \text{--- (6)}$$

Compare (1) & (6) we get

$$\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Hence verified

19) By using green's theorem evaluate $\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$ where C is the boundary in xy plane of the area enclosed by the x-axis and semi circle $x^2+y^2=1$; in the upper half of xyplane

Sol: $$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad \text{--- (1)}$$

$P = 2x^2 - y^2$; $Q = x^2+y^2$; $\frac{\partial P}{\partial y} = -2y$, $\frac{\partial Q}{\partial x} = 2x$

by using polar Coordinates

$x^2+y^2 = r^2$; $x = r\cos\theta$; $y = r\sin\theta$

$dx\,dy = r\,d\theta\,dr$

$r \Rightarrow 0$ to 1 $\qquad \theta \Rightarrow 0$ to π

$$\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy = \iint_R (2x+2y)\,dx\,dy$$

$$= \int_0^1 \int_0^{\pi} [2(r\cos\theta) + 2(r\sin\theta)]\,dx\,dy$$

$$= 2\int_0^1 r^2\,dr \cdot \int_0^{\pi} (\cos\theta + \sin\theta)\,d\theta$$

$$= 2\left[\frac{r^3}{3}\right]_0^1 \cdot [-\sin\theta + \cos\theta]_0^{\pi}$$

$$= 2\left(\frac{1}{3}\right)[(1+0) - (0-1)] \quad = 4/3$$

2) verify green's theorem in a plane for the integral $\int_C ((x-2y)dx + x\,dy)$ taken around the circle $x^2+y^2=1$

sol: $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ —— ①

$P = x-2y$; $Q = x$ $\qquad$ Given circle $x^2+y^2 = 1$

$\frac{\partial P}{\partial y} = -2$; $\frac{\partial Q}{\partial x} = 1$

by using polar Coordinates

$x^2+y^2 = r^2$

assume $x = r\cos\theta$; $y = r\sin\theta$ $\qquad r \Rightarrow 0$ to 1

$dx\,dy = r\,d\theta\,dr$ $\qquad \theta \Rightarrow 0$ to π

$$\int_C ((x-2y)dx + x\,dy) = \iint_R [1-(2)]\,r\,d\theta\,dr$$

$$= \int_0 \int_0^{\pi} 3r\,d\theta\,dr$$

$= 3\int_0^1 \int_0^{\pi} r\,dr\,d\theta$

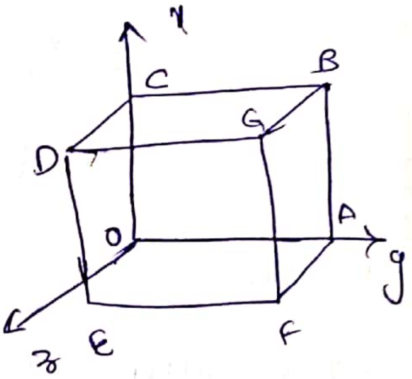
$= 3\int_0^1 r\,dr \int_0^{\pi} 1.d\theta$

$= 3\left[\frac{r^2}{2}\right]_0^1 . [\theta]_0^{\pi}$

$= 3(1/2)(2\pi)$

$= 3\pi.$

21. Verify Gauss divergence theorem for $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$ taken over the rectangular parallelopiped $0 \le x \le a$; $0 \le y \le b$; $0 \le z \le c$.



Sol: $\iint \vec{F}.\hat{n}\,ds = \iiint_V \nabla . \vec{F}\,dv.$

$\iiint_V \nabla . \vec{F}\,dv = \iiint \text{div}\,\vec{F}\,dv$

$\vec{F} = \cancel{d}\,(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$

$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$

$= 2x+2y+2z \qquad = 2(x+y+z)$

$\iiint \text{div}.\vec{F}\,dv = 2\int_0^c \int_0^b \int_0^a (x+y+z)\,dx\,dy\,dz$

$= 2\int_0^c \int_0^b \left(\frac{x^2}{2} + xy + xz\right)_0^a dy\,dz$

$= 2\int_0^c \int_0^b \left(\frac{a^2}{2} + ay + az\right) dy\,dz$

$= 2\int_0^c \left[\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right) dz$

$= \cancel{2}\frac{(abc)}{\cancel{2}}[a+b+c]$

$= abc\,[a+b+c] \quad \text{—} (1)$

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$$\iint \vec{F}\cdot\hat{n}\,ds = \underset{\substack{S_1\\ -\vec{k}}}{\iint} + \underset{\substack{S_2\\ \vec{k}}}{\iint} + \underset{\substack{S_3\\ -\vec{j}}}{\iint} + \underset{\substack{S_4\\ \vec{j}}}{\iint} + \underset{\substack{S_5\\ -\vec{i}}}{\iint} + \underset{\substack{S_6\\ \vec{i}}}{\iint}$$

S_1 at OABC :- $\hat{n} = -\vec{k}$

$$\underset{S_1}{\iint} \vec{F}\,\hat{n}\,ds = -\int_0^b\int_0^a (z^2 - xy)\,dx\,dy$$

$$= \int_0^b \left(\frac{x^2}{2}\right)_0^a y\,dy$$

$$= \left[\frac{a^2b^2}{4}\right]$$

at S_2 :- $\hat{n} = \vec{k}$ $z = c.$

$$\underset{S_2}{\iint} \vec{F}\,\hat{n}\,ds = \int_0^b\int_0^a (c^2 - xy)\,dx\,dy$$

$$= \int_0^b \left[c^2a - \frac{a^2}{2}y\right]dy$$

$$S_2 = \left[\frac{-a^2b^2}{4} + abc^2\right]$$

$\hat{n} = -j$ $y = 0$

S_3 :- $$\underset{S_3}{\iint} \vec{F}\,\hat{n}\,ds = \int_0^c\int_0^a (-zx)(-j)(j)\,dx\,dz$$

$$= \int_0^c \frac{za^2}{2}\,dz$$

$$S_3 = \frac{a^2c^2}{4}$$

S_4 :- $\hat{n} = \vec{j}$ $y = b$

$$\underset{S_4}{\iint} \vec{F}\,\hat{n}\,ds = \int_0^c\int_0^a (b^2 - zx)\,dx\,dy$$

$$= \int_0^c \left(ab^2 - \frac{za^2}{2}\right)dz$$

$$S_4 = \left[ab^2c - a^2c^2/4\right]$$

S_5 :- $\hat{n} = -\vec{i}$ $x = 0.$

$$S_5 = \frac{b^2c^2}{4}.$$

$S_6 \Rightarrow \hat{n} = \vec{i}$ $\quad x = a$

$$S_6 = \left[a^2bc - \frac{b^2c^2}{4}\right].$$

$$\iint_S \vec{F}\,\hat{n}\,ds = \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{a^2c^2}{4} + ab^2c - \frac{a^2c^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4}$$

$$= abc^2 + ab^2c + a^2bc$$

$$= abc\,[a+b+c] \quad -②$$

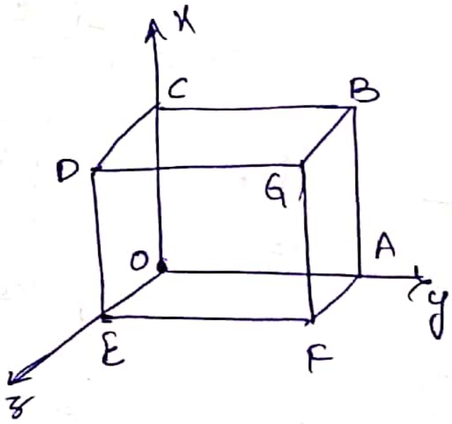
Since eqn ① & ② are equal

$$\iint_S \vec{F}\,\hat{n}\,ds = \iiint_S \nabla\cdot\vec{F}\,dv$$

Hence gauss divergence is verified

2d) verify Gauss divergence theorem for the vector function $\vec{F} = 4xz\vec{i} + y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x=0$; $y=0$; $z=0$ ~~at~~ and $x=1$, $y=1$, $z=1$

A $\quad \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$



$$\iiint_V \nabla\cdot\vec{F}\,dv = \iiint_V \text{div}\,\vec{F}\,dv.$$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$\iiint_V \text{div}\,\vec{F}\,dv = \int_0^1\!\int_0^1\!\int_0^1 (4z - xy)\,dx\,dy\,dz$$

$$= \int_0^1\!\int_0^1 [4xz - x\cdot y]_0^1\,dy\,dz$$

$$= \int_0^1\!\int_0^1 [4z - y]\,dy\,dz.$$

$$= \int_0^1 \left[4yz - \frac{y^2}{2}\right]_0^1 dz$$

$$= \left[4\frac{z^2}{2} - \frac{1}{2}z\right]_0^1$$

$$= 2 - \frac{1}{2}.$$

$$\iiint_V \operatorname{div}\vec{F}\, dv = \frac{3}{2} \quad \text{--- (1)}$$

$$\iint_S \vec{F}\hat{n}ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

S_1 at OAEF :- $\hat{n} = -\hat{i}$ $\quad x=0$

$$\iint_{S_1} \vec{F}\hat{n}ds = \int_0^1\int_0^1 (4xz)(-\vec{i})(\hat{i})\,dydz$$

$$= -\int_0^1\int_0^1 4xz\,dydz$$

$$= -\int_0^1 [4xyz]_0^1 dz$$

$$= x = 0 \Rightarrow S_1 = 0$$

S_2 at BCDG :- $\hat{n} = \vec{i}$ $\quad x = 1$

$$\iint_{S_2} \vec{F}\hat{n}ds = \int_0^1\int_0^1 4(1)z\,dydz$$

$$= \int_0^1 [4yz]_0^1 dz$$

$$= \int_0^1 (4z)dz = 2(1) = 2.$$

S_3 at OCDE :- $\hat{n} = -\hat{j}$ $\quad y=0$

$$\iint_{S_3} \vec{F}\cdot\hat{n}ds = \int_0^1\int_0^1 (-y^2)dxdz$$

$S_3 = 0$

S_4 at ABFG :- $\hat{n} = \vec{j}$; $y=1$

$$\iint_{S_4} \vec{F}\hat{n}ds = \int_0^1\int_0^1 (-1)\,dxdz$$

$$= -\int_0^1 [x]_0^1 dz$$

$$= -\int_0^1 1 \cdot dz$$

$$= -[z]_0^1$$

$S_4 = -1$

S_5 at OABC :- $\hat{n} = -\vec{k}$ $z = 0$.

$S_5 = 0$

S_6 at DEFG :- $\hat{n} = \vec{k}$; $z = 1$

$$S_6 = \int_0^1\int_0^1 y\,dx\,dy$$

$$= \int_0^1 [xy]_0^1\,dy = \int_0^1 y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^1$$

$$= 1/2.$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 0 + 2 + 0 - 1 + 0 + 1/2$$

$$= 1 + 1/2 = 3/2 \quad \text{——} \ (2)$$

eqⁿs (1) and (2) are equal.

Hence gauss divergence theorem is verified

23. verify gauss divergence theorem for $F = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the region by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

Sol: By divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\cdot dv.$$

$$\nabla\cdot\vec{F} = \left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right](4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})$$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 4 - 4y + 2z$$

$$4-4y+2z$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iiint_V (4-4y+2z)\,dx\,dy\,dz$$

Given $x^2+y^2=4 \Rightarrow y^2 = 4-x^2$

$$y = \pm\sqrt{4-x^2}$$

$x^2+y^2=r^2 \Rightarrow r=2 \Rightarrow x=\pm 2$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iiint (4-4y+2z)\,dx\,dy\,dz$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_0^3 (4-4y-2z)\,dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} \left(4z-4yz+2\frac{z^2}{2}\right]_0^3 dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} \left[12-12y+9\right] dy\,dx.$$

$$= \int_{-2}^{2} \left[21(\sqrt{4-x^2}) - 6(\sqrt{4-x^2})^2\right] - \left[21(-\sqrt{4-x^2}) - 6(-\sqrt{4-x^2})^2\right] dx$$

$$= 2\int_0^2 \left(21(\sqrt{4-x^2}) - 6(\sqrt{4-x^2})^2 + 21\sqrt{4-x^2} + 6(\sqrt{4-x^2})^2\right] dx$$

$$= 2\int_0^2 42\sqrt{4-x^2}\,dx$$

$$= 84\left[\int_0^2 \sqrt{4-x^2}\right]dx$$

$$\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right)$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 84\left[\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}(1)\right] = 84\left[\cancel{2}\pi/\cancel{2}\right] = 84\pi \quad (1)$$

Given $x^2+y^2=4$

$x=2\cos\theta$; $y=2\sin\theta$; $ds=2d\theta dz$

$$\hat{n}=\frac{\nabla\phi}{|\nabla\phi|} \qquad \phi=x^2+y^2-4.$$

$$\nabla\phi=\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)(x^2+y^2-4)$$

$$=2x+2y$$

$$|\nabla\phi|=\sqrt{4x^2+4y^2}=2\sqrt{x^2+y^2}=2(\sqrt{4})=2\times2=4.$$

$$\hat{n}=\frac{2x\vec{i}+2y\vec{j}}{4}=\frac{x}{2}\vec{i}+\frac{y}{2}\vec{j}$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds=\iint_{S_1}+\iint_{S_2}+\iint_{S_3}$$

$$\iint_{S_1}\vec{F}\cdot\hat{n}\,ds=\iint_{S_1}(4x\vec{i}+2y^2\vec{j}+z^2\vec{k})(-\vec{k})\,dxdy \qquad z=0,\ dz=0.$$

$S_1=0$

S_2 :- $\iint_{S_2}\vec{F}\cdot\hat{n}\,ds=\iint_{S_2}z^2\,dxdy$

$z=3$; $dz=0$; $\hat{n}=\vec{k}$

$$=9\iint_{S_2}dx\,dy$$

$$=9(4\pi)$$

$$=36\pi \quad \text{—— } (2)$$

S_3 :- $\iint_{S_3}\vec{F}\cdot\hat{n}\,ds=2\iint_{S_3}(4x\vec{i}-2y^2\vec{j}+z^2\vec{k})\left(\frac{x}{2}\vec{i}+\frac{y}{2}\vec{j}\right)dz\,d\theta.$

$$=2\int_0^{2\pi}\int_0^3\left(\frac{4x^2}{2}-\frac{2y^3}{2}\right)dz\,d\theta$$

$$=\int_0^{2\pi}\left[4x^2z-2y^3z\right]_0^3\,d\theta$$

$$\int_0^{2\pi} [12x^2 - 6y^3]\, d\theta$$

Sub $x = 2\cos\theta$; $y = 2\sin\theta$

$$= \int_0^{2\pi} 12(2\cos\theta)^2 - 6(2\sin\theta)^3 d\theta$$

$$= \int_0^{2\pi} [48\cos^2\theta - 48\sin^3\theta]\, d\theta$$

$$= 48\left[\frac{x}{2} + \frac{\sin 2x}{4} + c - \left\{\frac{\cos 3x}{3} + \cos x + c\right\}\right]_0^{2\pi}$$

$$= 48[\pi + c] - 48(0) + c$$

$$= 48\pi + c. \quad \text{———} (3)$$

Add (2) & (3) $\Rightarrow$ 84π

Hence Gauss divergence is verified.

24) using gauss divergence theorem $\iint_S \vec{F} \cdot \hat{n}\, ds$ where over the surface of $F = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and s is the surface of the sphere $x^2 + y^2 + z^2 = a^2$

Sol: $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

w.k.T $\iint_S \vec{F} \cdot ds = \iiint_V \nabla \cdot \vec{F}\, dv$

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(x^3 + y^3 + z^3)$$

$$= 3x^2 + 3y^2 + 3z^2$$

$$\iiint_V \nabla \cdot \vec{F}\, dv = 3\iiint (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

using spherical polar coordinates

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

$dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$; $x^2+y^2+z^2=r^2$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3(r^2)\left[r^2\sin\theta\, dr\,d\theta\,d\phi\right]$$

$$= 3\int_0^a r^4\,dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= 3\left[\frac{r^5}{5}\right]_0^a \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= \frac{3}{5}a^5\left(-(-1)-1\right)(2\pi)$$

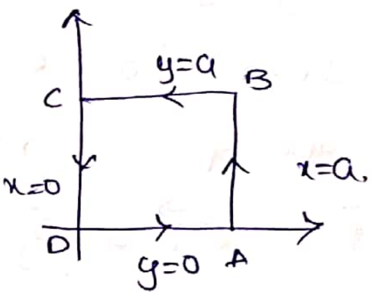
$$= \frac{3}{5}a^5(2)(2\pi)$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{12a^5\pi}{5}$$

25. verify stokes theorem for the function z=0 integrated roun[d] the square in the z=0 plane whose sides are along the lines $x=0$, $y=0$, $x=a$; $y=a$

Sol

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$



$$\oint_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$\therefore d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}.$$

$$\vec{F}\cdot d\vec{r} = \left[x^2\vec{i} + xy\vec{j}\right]\cdot\left[dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}\right]$$

$$= x^2dx + xy\,dy$$

Along OA :- $y=0$; $dy=0$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_{OA} x^2dx + xy\,dy$$

$$= \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

Along AB:- $x=a$; $dx=0$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{AB} x^2 dx + xy\,dy.$$

$$= \int_0^a ay\,dy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

Along BC:- $y=a$; $dy=0$.

$$\int_{BC} \vec{F}\,d\vec{r} = \int_{BC} x^2 dx + xy\,dy$$

$$= \int_a^0 x^2 dx = \left[\frac{x^3}{3}\right]_a^0 = -\frac{a^3}{3}.$$

Along CO:- $x=0$; $dx=0$

$$\int_{CO} \vec{F}\cdot d\vec{r} = \int_{CO} x^2 dx + xy\,dy$$

$$= 0$$

$$\oint_C \vec{F}\cdot d\vec{r} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$$

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\times\left(x^2\vec{i} + xy\,\vec{j}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ x^2 & xy & 0 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(xy)\right] - \vec{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= y\vec{k}$$

$$\iint_C \text{Curl } \vec{F}\cdot\hat{n}\,ds = \iint_S y\cdot\vec{k}\,\hat{n}\,ds$$

$$= \iint_R y\vec{k}\,\hat{n}\,\frac{dx\,dy}{\vec{k}\cdot\hat{n}}$$

$$= \iint_R y\,dx\,dy$$

$$= \iint_R y \cdot dx \cdot dy$$

$$= \int_0^a \int_0^a y \, dx \, dy$$

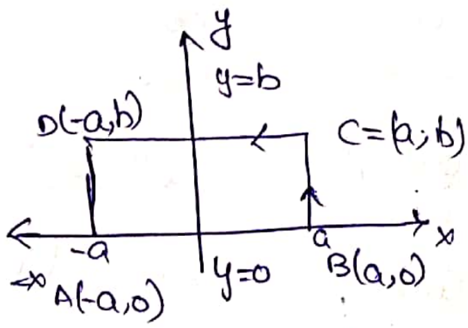
$$= \int_0^a [xy]_0^a dy$$

$$= \int_0^a ay \, dy = \frac{a \cdot a^2}{2} = \frac{a^3}{2}.$$

$$\therefore \oint_C \vec{F} \, d\vec{r} = \iint_C \text{Curl} \, \vec{F} \cdot \hat{n} \, ds$$

Hence proved

26 Verify Stokes theorem for a vector field $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region of xoy plane bounded the line $x = -a$; $x = a$; $y = 0$; $y = b$



Sol:- Stokes theorem:-

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl} \, \vec{F} \cdot \hat{n} \, ds.$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}.$$

$$\vec{F} \, d\vec{r} = [(x^2 - y^2)\vec{i} + 2xy\vec{j}] \cdot [dx\vec{i} + dy\vec{j} + dz\vec{k}]$$

$$= [x^2 - y^2] dx + 2xy \, dy$$

Along AB:- $y = 0$ $dy = 0$

$$\int_{AB} \vec{F} \, d\vec{r} = \int_{AB} (x^2 - y^2) dx + 2xy \, dy$$

$$= \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$= \frac{a^3}{3} - \left(\frac{-a^3}{3}\right)$$

$$= \frac{2a^3}{3}.$$

Along BC :- $x = a$; $dx = 0$

$$\int_{BC} \vec{F}\, d\vec{x} = \int_{BC} (x^2 - y^2)\,dx + 2xy\,dy$$

$$= \int_0^b 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_0^b = \frac{2ab^2}{2} = ab^2$$

Along CD:- $y = b$; $dy = 0$

$$\int_{CD} \vec{F}\cdot d\vec{x} = \int_{CD} (x^2 - y^2)\,dx + 2xy\,dy$$

$$= \int_{-a}^{a} (x^2 - b^2)\,dx = \left[\frac{x^3}{3} - xb^2\right]_a^{-a}$$

$$= \frac{-a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2$$

$$= \frac{-2a^3}{3} + 2ab^2$$

Along OA:- $x = -a$; $dx = 0$

$$\int_{OA} \vec{F}\cdot d\vec{x} = \int_{OA} (x^2 - y^2)\,dx + 2xy\,dy$$

$$= \int_b^0 -2ay\,dy = -2a\left[\frac{y^2}{2}\right]_b^0 = -2a\left(\frac{-b^2}{2}\right) = ab^2$$

$$\oint_C \vec{F}\, d\vec{x} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}.$$

$$= \frac{2a^3}{3} + ab^2 - \frac{2a^3}{3} + 2ab^2 + ab^2$$

$$= 4ab^2$$

Curl $\vec{F} = \nabla \times \vec{F} = \left(\vec{i}\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times \left[(x^2 - y^2)\vec{i} + 2xy\vec{j}\right]$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x^2 - y^2) & 2xy & 0 \end{vmatrix}$$

$= \vec{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2xy)\right] - \vec{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2-y^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2-y^2)\right]$

$= \vec{k}\,[2y+2y]$

$= 4y\,\vec{k}$

$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_S (4y)\,\vec{k}\,\hat{n}\,ds$

$= \iint_R (4y)\,\vec{k}\,\hat{n}\,\frac{dx\,dy}{|\vec{k}\,\hat{n}|}$

$= \iint_R 4y\,dx\,dy$

$= \int_0^b \int_{-a}^{a} 4y\,dx\,dy$

$= \int_0^b [4xy]_{-a}^{a}\,dy$

$= \int_0^b [4ay + 4ay]\,dy$

$= 8a\left(\frac{y^2}{2}\right)_0^b \Rightarrow \frac{8ab^2}{2} = 4ab^2$

$\therefore \oint_C \vec{F}\,d\vec{r} = \iint_S \text{curl}\,\vec{F}\,\hat{n}\,ds.$

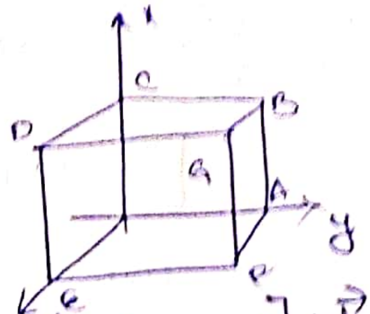
27. Verify Stoke's theorem for a vector field $\vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}$ where S is the surface of the cube $x=0, x=2, y=0, y=2; z=0, z=2$ above the xy-plane.

A $\oint \vec{F}\,d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

$\text{curl}\,\vec{F} = \nabla \times \vec{F}$

$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times \left[(y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}\right]$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y-z+2) & (yz+4) & -xz \end{vmatrix}$$



$$= \vec{i}\left[\frac{\partial}{\partial y}(-xz) - \frac{\partial}{\partial z}(yz+4)\right] + (-\vec{j})\left[\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial z}(y-z+2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(yz+4) - \frac{\partial}{\partial y}(y-z+2)\right].$$

$$= \vec{i}\,[-y] - \vec{j}\,[-z+1] + \vec{k}\,[-1].$$

$$\iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds = \iint_S (-y)\vec{i} + (1-z)\vec{j} + (-1)\vec{k}$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

S_1	S_2	S_3	S_4	S_5	S_6
$x=0$	$x=2$	$y=0$	$y=2$	$z=0$	$z=2$
$\hat{n}=-\vec{i}$	$\hat{n}=\vec{i}$	$\hat{n}=-\vec{j}$	$\hat{n}=\vec{j}$	$\hat{n}=-\vec{k}$	$\hat{n}=\vec{k}$

$$S_1 = \iint_{x=0} [-y]\vec{i} + (1-z)\vec{j} + (-1)\vec{k}\,]\,[-\vec{i}]\, dy\, dz$$

$$= \int_0^2\int_0^2 y\, dy\, dz = \int_0^2 \left.\frac{y^2}{2}\right]_0^2 dz$$

$$= \int_0^2 \left(\frac{4}{2}\right) dz = \int_0^2 2\, dz = 2z\Big]_0^2$$

$$= 4$$

$$S_2 = \iint_{x=2} [(-y)\vec{i} + (1-z)\vec{j} + (-1)\vec{k}\,]\,(\vec{i})\, dy\, dz$$

$$= \int_0^2\int_0^2 (-y)\, dy\, dz = \int_0^2 \left(\frac{y^2}{2}\right)_0^2 dz = -\int_0^2 2\, dz$$

$$= -4.$$

$$S_3 = \iint_{S_3} [(-y)\vec{i} + (1-z)\vec{j} + (-1)\vec{k}\,]\,(-\vec{j})\, dx\, dz$$

$$= \int_0^2 \int_0^2 [xz - x]_0^2\, dz$$

$$= \int_0^2 (2z-2)\,dz$$

$$= \left[\frac{2z^2}{2} - 2z\right]_0^2$$

$$= 4-4 = 0.$$

$$S_4 = \iint_{S_4} \left[(-y)\vec{i} + (1-z)\vec{j} + (-1)\vec{k}\right](\vec{j})\,dx\,dz.$$

$$= \int_0^2\int_0^2 (1-z)\,dx\,dz = \int_0^2 (x - zx)_0^2\,dz$$

$$= \int_0^2 (2-2z)\,dz = 4-4 = 0.$$

$$S_5 = \iint_{S_5} \left[(-y)\vec{i} + (1-z)\vec{j} + (-1)\vec{k}\right](-\vec{k})\,dx\,dy$$

$$= \int_0^2\int_0^2 dx\,dy = \int [x]_0^2\,dy = 2\int_0^2 dy = 4.$$

$$S_6 = \iint_{S_6} \left[(-y)\vec{i} + (1-z)\vec{j} + (-1)\vec{k}\right](\vec{k})\,dx\,dy$$

$$= \int_0^2\int_0^2 (-1)\,dx\,dy = \int_0^2 (-x)_0^2\,dy = 2\int_0^2 dy$$

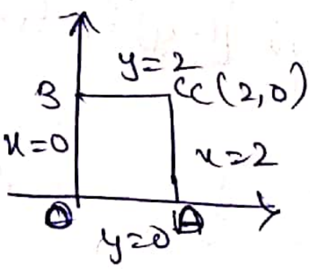
$$= 2\times 2 = 4.$$

$$\therefore \iint_S \operatorname{Curl}\vec{F}\cdot\hat{n}\,ds = 4-4+4-4$$

$$= 0.$$

$$\int_C = \int_{OA} + \int_{AC} + \int_{CB} + \int_{BO}$$

OA	AC	CB	BO
$y=0$	$x=2$	$y=2$	$x=0$
$dy=0$	$dx=0$	$dy=0$	$dx=0$



$$\int_C \vec{F}\cdot d\vec{x} = \int_{OACB} (y-z+2)\,dx + (yz+4)\,dy + (-xz)\,dz$$

$z=0$ at xoy plane.

At OA :- $y=0$; $dy=0$

$$\int_{OA} \vec{F}\, d\vec{r} = \int_{OA} (y-z+2)dx + (yz+4)dy + (-xz)dz$$

$$= \int_0^2 2\,dx = 2[x]_0^2 = 4.$$

At AC :- $x=2$ $dx=0$

$$\int_{AC} \vec{F}\, d\vec{r} = \int_{AC} (y+2)dx + 4dy.$$

$$= \int_0^2 4dy = 4[y]_0^2 = 8$$

At CB :- $y=0$; $dy=0$

$$\int_{CB} \vec{F}\, d\vec{r} = \int_{CB} (y+2)dx + 4dy$$

$$= \int_0^2 2dx + 2[-x]_2^0 = 2(-2) = -4.$$

At BO :- $x=0$ $dx=0$

$$\int_{BO} \vec{F} \cdot d\vec{r} = \int_{BO} (y+2)dx + 4dy$$

$$= \int_2^0 4dy = 4[y]_2^0 = -8$$

$$\oint_C \vec{F} \cdot dr = 4+8+-4-8 = 0.$$

$$\therefore \oint \vec{F}\, d\vec{r} = \iint_S \text{Curl}\, \vec{F} \cdot \hat{n}\, ds.$$

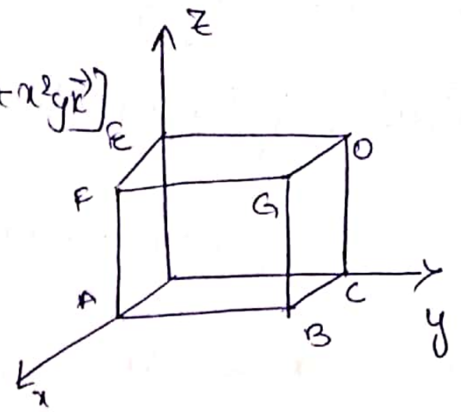
18) Verify stoke's theorem for $\vec{F} = y^2 z\,\vec{i} + z^2 x\,\vec{j} + x^2 y\,\vec{k}$ where s is the open surface of the cube formed by the planes $x=-a$, $x=a$, $y=-a$, $y=a$, $z=-a$; $z=a$ in which $z=-a$ is cut open.

Sol: Stokes theorem,

$$\oint \vec{F}\, d\vec{r} = \iint \text{Curl}\, \vec{F} \cdot \hat{n}\, ds$$

$\text{Curl}\,\vec{F} = \nabla \times \vec{F}$

$= \left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right] \times \left[y^2 z\,\vec{i} + z^2 x\,\vec{j} + x^2 y\,\vec{k}\right]$



$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$

$= \vec{i}\left[\frac{\partial}{\partial y}(x^2 y) - \frac{\partial}{\partial z}(z^2 x)\right] - \vec{j}\left[\frac{\partial}{\partial x}(x^2 y) - \frac{\partial}{\partial z}(y^2 z)\right] + \vec{k}\left[\frac{\partial}{\partial x}(z^2 x) - \frac{\partial}{\partial y}(y^2 z)\right]$

$= \vec{i}\,[x^2 - 2xz] - \vec{j}\,[2xy - y^2] + \vec{k}\,[z^2 - 2yz]$

$\iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$ $\quad S_6$ is open

S_1: $x = -a$, $\hat{n} = -\vec{i}$; S_2: $x = a$, $\hat{n} = \vec{i}$; S_3: $y = -a$, $\hat{n} = -\vec{j}$; S_4: $y = a$, $\hat{n} = \vec{j}$; S_5: $z = -a$, $\hat{n} = -\vec{k}$

$\iint_{S_1} = \iint_{S_1} \left[(x^2 - 2xz)\vec{i} + [y^2 - 2xy]\vec{j} + [z^2 - 2yz]\vec{k}\right](-\vec{i})\,dy\,dz$

$= \int_{-a}^{a}\int_{-a}^{a} [2xz - x^2]\,dy\,dz = \int_{-a}^{a}\int_{-a}^{a} [-2az - a^2]\,dy\,dz$

$= \int_{-a}^{a} [-2ayz - ya^2]_{-a}^{a}\,dz = \int_{-a}^{a} -2a\,[2az + a^2]\,dz$

$= -2a\left[2a.\frac{z^2}{2} + a^2 z\right]_{-a}^{a}$

$= -4a^4.$

$\iint_{S_2} \left[(x^2 - 2xz)\vec{i} + (y^2 - 2xy)\vec{j} + (z^2 - 2yz)\vec{k}\right](\vec{i})\,dy\,dz$

$= \iint_{-a\,-a}^{a\;\;a} (2xz - x^2)\,dy\,dz = \int_{-a}^{a} (-2ayz - ya^2)_{-a}^{a}\,dz$

$= \int_{-a}^{a} -2a\,(2az + a^2)\,dz \quad = 4a^4$

$$S_3 = \iint_{S_3} (2xy - y^2)\,dx\,dz = \int_{-a}^{a}\int_{-a}^{a} (-2ax - a^2)\,dx\,dz$$

$$= \int_{-a}^{a} \left[-\frac{2ax^2}{2} - ax\right]_{-a}^{a} dz = \int_{-a}^{a} (-2a^3)\,dz = -2a^3(2a)$$

$$= -4a^4$$

$$S_4 = \iint_{S_4} [y^2 - 2xy]\,dx\,dz = \int_{-a}^{a}\int_{-a}^{a} (a^2 - 2ax)\,dx\,dz$$

$$= \int_{-a}^{a} \left[a^2 x - 2a\frac{x^2}{2}\right]_{-a}^{a} dz = \int_{-a}^{a} 2a^3\,dz = 2a^3 [z]_{-a}^{a}$$

$$= 4a^4.$$

$$S_5 = \int_{-a}^{a}\int_{-a}^{a} (z^2 - 2yz)\,dx\,dy.$$

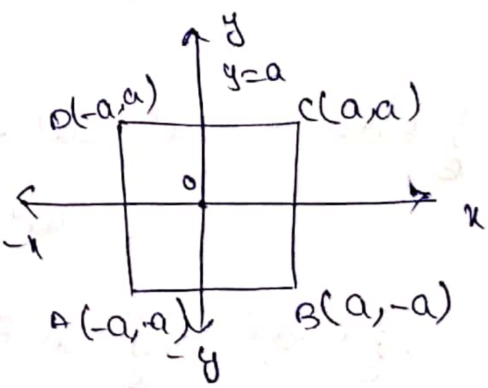
$$= \int_{-a}^{a} 2a[\,a^2 - 2ay\,]\,dy$$

$$= 2a\left[a^2 y - 2a\frac{y^3}{2}\right)_{-a}^{a} = 4a^4.$$

$$\iint_S \text{Curl}\,\vec{F} \cdot \hat{n}\,ds = -4a^4 + 4a^4 - 4a^4 + 4a^4 + 4a^4.$$

$$= 4a^4.$$

$$\oint_C = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$



$$\int_C \vec{F} \cdot d\vec{r} = \int_{ABCD} [(y^2 z)\,dx + (z^2 x)\,dy + (x^2 y)\,dz] \;-(1)$$

$z = -a$, $dz = 0$

$$= \int_{ABCD} (-ay^2)\,dx + (a^2 x)\,dy$$

Along AB:- $y = -a$; $dy = 0$.

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_{AB} (-ay^2)\,dx + (a^2 x)\,dy = \int_{-a}^{a} -a^3\,dx = -a^3 [x]_{-a}^{a}$$

$= -2a^4.$

Along BC :- $x = a$; $dx = 0$

$$\int_{BC} \vec{F} \cdot \vec{dx} = \int_{BC} (-ay^2)\,dx + (a^2 x)\,dy$$

$$= \int_{-a}^{a} a^3\,dy$$

$$= a^3 (2a) = 2a^4.$$

Along CD :- $y = 0$; $dy = 0$

$$\int_{CD} \vec{F} \cdot \vec{dx} = \int_{CD} (-ay^2)\,dx + (a^2 x)\,dy$$

$$= \int_{+a}^{-a} -a^3\,dx = -a^3[-a-a] = 2a^4$$

Along DA :- $x = -a$; $dx = 0$

$$\int_{DA} \vec{F} \cdot \vec{dx} = \int_{DA} (-ay^2)\,dx + (a^2 x)\,dy$$

$$= \int_{a}^{-a} (-a^3)\,dy = -a^3 [y]_a^{-a}$$

$$= +2a^4.$$

$$\int_{c} \vec{F} \cdot \vec{dx} = 2a^4 + 2a^4 - \cancel{2a^4} + \cancel{2a^4}$$

$$= 4a^4.$$

$$\therefore \oint_C \vec{F} \cdot \vec{dx} = \iint_S \text{Curl } \vec{F} \cdot \hat{n}\,ds.$$

Hence proved.

Unit III: Special Functions –I

Unit – III

1) Define ordinary point (K1)
2) Define singular point (K2)
3) When we call a singular point is regular? (K1)
4) Define irregular singular point (K1)
5) Define recurrence relation (K2)
6) Define indicial equation on series solution when x=0 is a regular singularity (K1)
7) Write the complete solution when roots of the indicial equation are distinct. (K1)
8) Write the complete solution when roots of the indicial equation are equal. (K2)
9) Write the complete solution when roots of indicial equation are distinct and differ by an integer making a co-efficient of y infinites (K1)
10) Write the Bessel's equation of order n (K1)
11) Write the Bessel's equation of order zero (K1)
12) Write the Neumann function. (K2)
13) Write Bessel function of the second kind of order n (K1)
14) Write any two recurrence formula for $J_n(x)$ (K1)
15) Write the expansion for J_0 (K2)
16) Write the expansion for J_1 (K1)
17) Write the value of $J_{1/2}$ (K1)
18) Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$ (K2)
19) Reduce the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(k^2x^2-n^2)y=0$ to Bessel function. (K1)
20) Reduce the differential equation $x^2\frac{d^2y}{dx^2}+a\frac{dy}{dx}+k^2xy=0$ to Bessel function. (K1)

Assignment - 5.

unit - 3 (2 marks)

1. Define ordinary point?

A. For a differential equations $P_0(x) \cdot \frac{d^2y}{dx^2} + P_1 x \frac{dy}{dx} + P_2(x) y = 0$ if $P_0(a) \neq 0$, then $x = a$ is called as an ordinary point of the differential equation.

2. Define singular point?

A. For differential equation $P_0(x) \frac{d^2y}{dx^2} + P_1(x) \frac{dy}{dx} + P_2(x) y = 0$, if $P_0(a) = 0$, then $x = a$ is called as singular point.

3. When we can a singular point is Regular?

A. A point x_0 is regular singular point if the functions $(x - x_0) p(x)$ and $(x - x_0)^2 Q(x)$ are both analytic at x_0 otherwise x_0 is irregular.

4. Define irregular singular point?

A. If the function $(x - x_0) p(x)$ and $(x - x_0)^2 Q(x)$ are not analytic at x_0, then x_0 is irregular singular point if $p(x_0) = 0$

5. Define Reccurrence relation?

A. A Reccurrence relation is an equation which represents a sequence based on some results. It helps in finding the subsequent from dependent upon the preceeding team.

6. Define indical equation - on series solution when $x = 0$ is a regular singularity?

A. If $x = 0$ is a regular singularity point, if the normalize differential equation $y'' + p(x)y' + q(x)y$ is such that $xp(x)$ and $x^2q(x)$ are analytic at $x=0$. Then, the quadratic equation obtained by equating the coefficient of lowest degree terms in x to zero of its solution is known as the indicial equation.

7. Write the complete solution when roots of the indicial equation are distinct?

A. When the roots of indicial equation are distinct and doesnot differ by an integer, then complete solution is $y = C_1(y)_{m_1} + C_2(y)_{m_2}$, where m_1, m_2 are roots.

8. Write the complete solution when roots of the indicial equations are equal?

A. When roots of the indicial equations are equal, the complete solution is,

$y = C_1(y)_{m_1} + C_2\left(\frac{\partial y}{\partial m}\right)_{m_2}$ where m_1, m_2 are roots.

9. Write the complete solution when roots of indicial equations are distinct and differ by an integer making coefficient of y- infinities?

A. The complete solution is

$y = C_1(y)_{m_2} + C_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$, where m_1, m_2 are roots such that $m_1 < m_2$.

10. Write the Bessel's equation of order n?

A. The Bessel's equation of order n is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\,y = 0 \qquad n \geq 0$$

11. Write the Bessel's equation of order zero?

A. The Bessel's equation of order zero is

$$x^2 \frac{d^2y}{dx^2} + y\,x\,\frac{dy}{dx} + x^2 y = 0.$$

12. Write the Neumann function?

A. Neuman function is also called as the Bessel's function of the second kind of order n. It is denoted by $Y_n(x)$

$$y_n(x) = J_n(x) \frac{dx}{x[J_n(x)]^2}$$

13. Write Bessel's function of second kind of order n?

A) When n is integral, another independent integral of bessel's function is needed to form its general equation.

$$y_n(x) = J_n(x) \int \frac{dx}{x[J_n(x)]^2}$$

$Y_n(x)$ is called as second kind of order n.

Bessel's function (or) Neumann function.

4) Write two Reccurence formula for $J_n(x)$

The Reccurence formula for $J_n(x)$

* $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x).$

* $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

15. Write the expansion for J_0?

A. $J_0(x) = 1 - \frac{1}{1!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!}\left(\frac{x}{2}\right)^6 + \cdots$

16. Write the expansion for J_1?

A. $J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!4!}\left(\frac{x}{2}\right)^6 + \cdots\right]$.

17. Write the value of $J_{1/2}$?

Ans: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$.

Put $n = 1/2$

$J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{\pi x}}\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right] = \sqrt{\frac{2}{\pi x}}\sin x$

18. Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.

A. $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

Put $n = 1, 2, 3, 4,$ in $J_{n+1}(x)$

$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$

$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$

$J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$

$J_5(x) = \frac{8}{x} J_4(x) - J_3(x)$

Sub $J_2(x)$ in $J_3(x)$

$J_3(x) = \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$.

$J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x) \longrightarrow (1)$

Now put $J_3(x)$ and $J_2(x)$ in $J_4(x)$

$J_4(x) = \left[\frac{48}{x^3} - \frac{8}{x}\right] J_1(x) + \left[1 - \frac{24}{x^2}\right] J_0(x) \quad — (2)$

Now put ① and ② in $J_5(x)$ we get

$$J_5(x) = \left[\frac{384}{x^4} - \frac{72}{x^2} - 1\right] J_1(x) + \left[\frac{12}{x} - \frac{92}{x^3}\right] J_0(x) \text{ //.}$$

Q. Reduce the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - n^2)y = 0$ to bessel function?

Ans. To reduce differential eq

$$x^2\frac{d^2y}{dx^2} + x.\frac{dy}{dx} + (k^2x^2 - n^2)y = 0 \text{ —— ①}$$ to bessel form

Put $t = kx$, so that $\frac{dy}{dx} = k\frac{dy}{dt}$ & $\frac{d^2y}{dx^2} = k^2\frac{d^2y}{dt^2}$

Now ① becomes $t^2\frac{d^2y}{dx^2} + t.\frac{dy}{dt} + (t^2 - n^2)y = 0$ is the required bessel function.

20. Reduce the differential eqn $\frac{d^2y}{dx^2}x^2 + a.\frac{dy}{dx} + k^2 xy = 0$ to bessel's function.

A. put $y = x^n z$

$$\frac{dy}{dz} = x^n.\frac{dz}{dx} + n.x^{n-1}z.$$

$$\frac{d^2y}{dx^2} = x^n\frac{d^2z}{dx^2} + 2n\,x^{n-1}\frac{dz}{dx} + [n(n-1)\,x^{n-2}z]$$

The equation becomes

$$x^{n+1}\frac{d^2z}{dx^2} + (2n+a)x^n\frac{dz}{dx} + [k^2x^2 + n^2 + (a-1)n]\,x^{n-1}z = 0$$

Dividing by x^{n-1} and putting $2n+a=1$

$$x^2\frac{d^2z}{dx^2} + x.\frac{dz}{dx} + (k^2x^2 - n^2)z = 0.$$

SPECIAL FUNCTION -I

1 Solve in series the equation $\frac{d^2y}{dx^2} + xy = 0$ (K3)

2 Solve in series $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$ [redacted]

3 Solve in series the equation $9x\left(1-x\right)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$ (K4)

4 Solve in series the equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$ (K4)

5 Obtain the series solution of the equation $x\left(1-x\right)\frac{d^2y}{dx^2} - \left(1+3x\right)\frac{dy}{dx} - y = 0.$ (K5)

6 Solve in series $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2-4\right)y = 0.$ (K5)

7 Solve in series $xy'' + 2y' + xy = 0.$ (K5)

(OR)

8.Prove that $J_{\frac{5}{2}}\left(x\right) = \sqrt{\frac{2}{\pi x}}\left\{\frac{\left(3-x^2\right)}{x^2\sin x} - \frac{3}{x}\cos x\right\}.$ (K6)

9.Prove that $J_n''\left(x\right) = \frac{1}{4}\left[J_{n-2}\left(x\right) - 2J_n\left(x\right) + J_{n+2}\left(x\right)\right].$ (K5)

10. Prove that $\frac{d}{dx}\left[x\,J_n\left(x\right)J_{n+1}\left(x\right)\right] = x\left[J_n^2\left(x\right) - J^2_{n+1}\left(x\right)\right]$ (K5)

11. Prove that

$$a)\int J_3\left(x\right)dx = C - J_2\left(x\right) - 2/xJ_1\left(x\right).$$

$$b)\int x\,J_0^2\left(x\right)dx = \frac{1}{2}x^2\left[J_0^2\left(x\right) + J_1^2\left(x\right)\right].$$

(K6)

12. Show that

$$\text{a) } \mathrm{J_n}\left(\mathrm{x}\right) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)d\theta, \text{ n integer}$$

$$(b)\ \mathrm{J_0}\left(\mathrm{x}\right) = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\phi)d\phi$$

(K5)

13.Show that $J_0^2 + 2J_1^2 + 2.J^2{}_2 + \ldots\ldots = 1.$ (K4)

14. Reduce the differential equation $x\frac{d^2y}{dx^2} + c\frac{dy}{dx} + k^2x^ry = 0$ (K5)

to Bessel form by putting $x = t^m$.

15. Solve the differential equations a) $y'' + \frac{y'}{x} + \left(8 - \frac{1}{x^2}\right)y = 0.$ (K5)

b) $4y'' + 9xy = 0$. (K5)

16. Solve the differential equation $xy'' + y' + \frac{1}{4}y = 0$ (K4)

17. Explain the orthogonality of Bessel function. (K3)

18. If $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots, \alpha_n$ are the positive roots of $J_0(x) = 0$,

Show that $\frac{1}{2} = \sum_{n=1}^{\infty}\left[\frac{J_0(\alpha_n x)}{\alpha_n J_1(\alpha_n)}\right]$. (K6)

19. Expand $f(x) = x^2$ in the interval $0 < x < 2$ interms of $J_2(\alpha_n x)$, where α_n are determined by $J_2(2\,\alpha_n) = 0$. (K4)

Assignment - 6.

Unit - III

1) Solve in series the equation $\frac{d^2y}{dx^2} + xy = 0$.

Sol: $x=0$ is an ordinary point ; Coefficient of $\frac{d^2y}{dx^2} \neq 0$ at $x=0$; $\frac{d^2y}{dx^2} + xy = 0$ ——— (1)

Its solution is

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots + a_nx^n + \ldots \quad \text{——— (2)}$$

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \ldots\ldots + na_nx^{n-1} + \ldots\ldots$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + \ldots\ldots + n(n-1)a_nx^{n-2} + \ldots$$

Sub $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1)

$$2a_2 + 6a_3x + 12a_4x^2 + \ldots\ldots + n(n-1)a_nx^{n-2} + \ldots + x(a_0 + a_1x + \ldots + a_nx^n) = 0.$$

$$2a_2 + (6a_3 + a_0)x + (12a_4 + a_1)x^2 + (20a_5 + a_2)x^3 + \ldots\ldots + [(n+2)(n+1)a_{n+2} + a_{n-1}]x^n + \ldots = 0.$$

Coefficients of various powers of x^n equating to zero.

Coefficients of Constant $(x^0) = 0$.

$a_2 = 0$.

Coefficient of $x = 0$

$6a_3 + a_0 = 0$

$a_3 = \frac{-a_0}{6}.$

Coefficient of $x^2 = 0$.

$12a_4 + a_1 = 0$

Coefficient of $x^3 = 0$

$$20a_5 + a_2 = 0$$

$$a_5 = \frac{-a_2}{20} = 0 \qquad \{\because a_2 = 0\}$$

Coefficient of $x^n = 0$

$$(n+2)(n+1)\,a_{n+2} + a_{n-1} = 0$$

$$a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \quad \text{——— (3)}$$

from (3)

$$a_6 = \frac{-a_3}{6\cdot 5} = \frac{a_0}{120}$$

$$a_7 = \frac{-a_4}{7\cdot 6} = \frac{a_1}{504}$$

$$a_8 = \frac{-a_5}{8\cdot 7} = 0$$

$$a_9 = \frac{-a_6}{9\cdot 8} = \frac{-a_0}{8640}$$

Sub in (2), the required solution is

$$y = a_0 + a_1 x + \left(\frac{-a_0}{6}\right)x^3 + \left(\frac{-a_1}{21}\right)x^4 + \frac{a_0}{120}x^6 + \frac{a_1}{504}x^7 + $$

$$-\left(\frac{a_0}{8640}\right)x^9 + \ldots$$

$$y = a_0\left[1 - \frac{x^3}{6} + \frac{x^6}{120} - \frac{x^9}{8640} + \ldots\right] + a_1\left[x - \frac{x^4}{12} + \frac{x^7}{504} + \ldots\right]$$

2. Solve in series $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$

Sol: $x=0$ is an ordinary point since Coefficient of $\frac{d^2y}{dx^2} \neq 0$ when $x=0$

The Solution is

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots + n(n-1)a_nx^{n-2} + \cdots$$

Sub $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in given equation.

$$(1-x^2)\left[2a_2 + 6a_3x + 12a_4x^2 + \cdots + (n)(n-1)a_nx^{n-2} + \cdots\right] - x\left[a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots\right] + 4\left[a_0 + a_1x + \cdots\right] = 0.$$

Equating coefficient of various powers of x to zero.

$2a_2 + 4a_0 = 0$ $\{$coefficient of $x^0 = 0\}$.

$a_2 = -2a_0$

$6a_3 - a_1 + 4a_1 = 0$

$a_3 = \frac{-a_1}{2}$ $\{$Coefficient of $x = 0\}$

$12a_4 - 2a_2 - 2a_2 + 4a_2 = 0$ $\{$Coefficient of $x^2 = 0\}$

$a_4 = 0$

$20a_5 - 6a_3 - 3a_3 + 4a_3 = 0$ $\{$Coefficient of $x^3 = 0\}$

$20a_5 - 5a_3 = 0$

$a_5 = \frac{a_3}{4} = \frac{-a_1}{8}$.

$(n+2)(n+1)a_{n+2} - n(n+1)a_n - na_n + 4a_n = 0$

$\{$Coefficient of $x^n = 0\}$

$$a_{n+2} = \frac{n-2}{n+1} a_n$$

Put $n = 4, 5, 6, 7$

$a_6 = \frac{2}{5} a_4 = 0$;

$a_7 = \frac{3}{6} a_5 = \frac{-a_1}{16}$

$a_8 = \frac{4}{7} a_6 = 0$;

$a_9 = \frac{5}{8} a_7 = \frac{-5a_1}{128}$

Sub in y, we get required solution..

$$y = a_0 + a_1 x + (-2a_0) x^2 + \left(\frac{-a_1}{2}\right) x^3 + \left(\frac{-a_1}{8}\right) x^5 + \left(\frac{-a_1}{16}\right) x^7 + \left(\frac{-5a_1}{128}\right) x^9 + \cdots$$

$$y = a_0 (1 - 2x^2) + a_1 x \left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} - \cdots\right)$$

3. Solve in series the equation $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$

Sol:- $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$ —— (1)

$x = 0$ is singular point since Coefficients of $\frac{d^2y}{dx^2} = 0$ at $x = 0$

The solution is

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$$

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + \cdots$$

$$\frac{d^2y}{dx^2} = m(m-1) a_0 x^{m-2} + m(m+1) a_1 x^{m-1} + (m+1)(m+2) a_2 x^m + \cdots$$

Sub $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1)

$9x(1-x)\left[m(m-1)a_0x^{m-2}+m(m+1)a_1x^{m-1}+\dots\right]-12\left[ma_0x^{m-1}+\right.$

$\left.(m+1)a_1x^m+(m+2)a_2x^{m+1}+\dots\right]+4\left[a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\dots\right]=0$

Equating Coefficient of least power of x to zero i.e., x^{m-1}

$9m(m-1)a_0-12ma_0=0$

$m(3m-7)=0 \Rightarrow m=0$ (or) $m=7/3$

The roots of indicial equation are $0, 7/3$.

They are distinct and donot differ by integer coe

Coefficient of $x^m=0$

$9m(m+1)a_1-9m(m-1)a_0-12(m+1)a_1+4a_0=0$

$a_1(9m(m+1)-12(m+1))+a_0(4-9m(m-1))=0$

$3a_1(3m-4)(m+1)-a_0(3m-4)(3m+1)=0$

$3a_1(m+1)=a_0(3m+1)$

$a_1=\dfrac{3m+1}{3(m+1)}a_0$

Coefficient of $x^{m+1}=0$

$-9m(m+1)a_1-12(m+2)a_2+4a_1+9(m+1)(m+2)a_2=0$

$\left[4-9m(m+1)\right]a_1+a_2\left[9(m+1)(m+2)-12(m+2)\right]=0$

$-(3m+4)(3m-1)a_1+3a_2(m+2)(3m-4)=0$

$3a_2(m+2)=a_1(3m+4) \Rightarrow a_2=\dfrac{(3m+4)(3m+1)}{3(m+2)\,3(m+1)}a_0$

Coefficient of $x^{m+2}=0$

$-9(m+1)(m+2)a_2+9(m+3)(m+2)a_3-12(m+3)a_3+4a_2=0$

$$a_3 = \frac{(3m+7)(3m+4)(3m+1)}{27(m+3)(m+2)(m+1)} a_0$$

When $m=0$

$$a_1 = \frac{1}{3} a_0$$

$$a_2 = \frac{4}{18} a_0 = \frac{2}{9} a_0$$

$$a_3 = \frac{14}{81} a_0.$$

$$y_1 = a_0 \left[1 + \frac{1}{3}x + \frac{2}{9}x^2 + \frac{14}{81}x^3 + \cdots\right]$$

when $m = 7/3$

$$y_2 = a_0 x^{7/3} \left[1 + \frac{8}{10}x + \frac{88}{130}x^2 + \frac{616}{1040}x^3 + \cdots\right]$$

The Complete solution is

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 a_0 \left[1 + \frac{1}{3}x + \frac{2}{9}x^2 + \frac{14}{81}x^3 + \cdots\right] + C_2 a_0 x^{7/3}\left[1 + \frac{8}{10}x + \frac{88}{130}x^2 + \frac{616}{1040}x^3 + \cdots\right]$$

4. Solve in series the equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$

Sol: $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$ —— (1)

$x=0$ is singular point. the Coefficient of $\frac{d^2y}{dx^2} = 0$ at $x=0$

The solution is

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3}$$

$$\frac{dy}{dx} = ma_0 x^{m-1} + (m+1)a_1 x^m + (m+2)a_2 x^{m+1} + (m+3)a_3 x^{m+2}$$

$$\frac{d^2y}{dx^2} = m(m-1)a_0 x^{m-2} + m(m+1)a_1 x^{m-1} + (m+1)(m+2)a_2 x^m + \dots$$

Sub $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in ①

$$x[m(m-1)a_0 x^{m-2} + m(m+1)a_1 x^{m-1} + \dots] + [ma_0 x^{m-1} + (m+1)a_1 x^m + \dots] +$$

$$x[a_0 x^m + a_1 x^{m+1} + \dots] = 0$$

lowest power of x to zero (x^{m-1}):-

$$a_0(m)(m-1) + ma_0 = 0$$

$$a_0 m^2 = 0 \Rightarrow m^2 = 0$$

$$\Rightarrow m = 0. \quad \text{(the roots are identical)}$$

Equating Coefficient of $x^m, x^{m+1}, x^{m+2}, \dots$ to zero

$$a_1(m)(m+1) + (m+1)a_1 = 0 \qquad \text{(Coefficient of } x^m)$$

$$a_1(m+1)(m+1) = 0$$

$$a_1 = 0$$

$$a_2(m+1)(m+2) + (m+2)a_2 + a_0 = 0 \qquad \text{(coefficient of } x^{m+1})$$

$$a_2[m+1+1](m+2) + a_0 = 0$$

$$a_2 = \frac{-a_0}{(m+2)^2}$$

$$a_1 + (m+3)a_3 + (m+2)(m+3)a_3 = 0 \qquad \text{(coefficient of } x^{m+2})$$

$$a_3 = 0$$

$$a_2 + (m+4)a_4 + (m+4)(m+3)a_4 = 0.$$

(coefficient of x^{m+3})

$$a_4 = \frac{-a_2}{(m+4)^2} = \frac{a_0}{(m+2)^2(m+4)^2}$$

$$y = a_0 x^m \left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \cdots\right] \quad \text{--- (2)}$$

put $m=0$, the first solution is

$$y_1 = a_0\left[1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots\right]$$

Since the roots are identical to get second solution,

Partial differentiate (2) w.r.t m.

$$\frac{\partial y}{\partial m} = y\log x + a_0 x^m \left[\frac{x^2}{(m+2)^2}\left(\frac{2}{m+2}\right) - \frac{x^4}{(m+2)^2(m+4)^2}\left[\frac{2}{m+2} + \frac{2}{m+4}\right] + \cdots\right]$$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0}$$

$$= y_1 \log x + a_0\left[\frac{x^2}{4} - \frac{3x^4}{128} + \cdots\right]$$

The Complete solution is $y = c_1 y_1 + c_2 y_2$

$$y = c_1 a_0\left[1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots\right] + c_2 y_1 \log x + c_2 a_0\left[\frac{x^2}{4} - \frac{3x^4}{128} + \cdots\right].$$

$$y = (c_1 a_0 + c_2 \log x)\left[1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots\right] + c_2 a_0\left[\frac{x^2}{4} - \frac{3x^4}{128} + \cdots\right]$$

5 Obtain the series solution of the equation $x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0.$

Sol: $x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0$ --- (1)

$x=0$ is singular point, since Coefficient of $\frac{d^2y}{dx^2} = 0$ at $x=0$

The solution is

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$$

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + \cdots$$

$$\frac{d^2y}{dx^2} = m(m-1) a_0 x^{m-2} + (m+1)(m) a_1 x^{m-1} + (m+1)(m+2) a_2 x^m + \cdots$$

Sub $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$.

$$x(1-x)\left[m(m-1) a_0 x^{m-2} + m(m+1) a_1 x^{m-1} + \cdots\right] - (1+3x)\left[m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + \cdots\right] - \left[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots\right] = 0$$

Equating coefficient of least power of x (x^{m-1}) to zero.

$m(m-1) a_0 - m a_0 = 0 \Rightarrow m(m-2) = 0$

$m = 0$; $m = 2$;

The roots are distinct and differ by an integer. Equating to zero the coefficient of various powers of x :-

Coefficient of $x^m = 0$

$$m(m+1) a_1 - m(m-1) a_0 - (m+1) a_1 - 3m a_0 - a_0 = 0$$

$$a_1 (m+1)(m-1) = a_0 (m(m-1) + 3m + 1)$$

$$a_1 (m+1)(m-1) = a_0 (m^2 + 2m + 1)$$

$$a_1 (m-1) = a_0 (m+1) \Rightarrow a_1 = (a_0)\left(\frac{m+1}{m-1}\right)$$

Coefficient of $x^{m+1} = 0$.

$$-m(m+1) a_1 + (m+1)(m+2) a_2 - (m+2) a_2 - 3(m+1) a_1 - a_1 = 0$$

$$a_2 (m+2)(m) = a_1 (m(m+1) + 3m + 3 + 1)$$

$$a_2(m+2)(m) = a_1(m^2+4m+4)$$

$$a_2(m) = a_1(m+2) \Rightarrow a_2 = \frac{m+2}{m} a_1$$

$$a_2 = \frac{(m+2)(m+1)}{m(m-1)} a_0$$

Coefficient of $x^{m+2} = 0$.

$$-(m+1)(m+2)a_2 + (m+2)(m+3)a_3 - (m+3)a_3 - 3(m+2)a_2 - a_2 = 0.$$

$$a_3(m+3)(m+1) = a_2((m+1)(m+2)+(m+3) + m+2-1)$$

$$a_3 = \frac{a_2(m+3)}{m+1} = \frac{(m+3)(m+2)(m+1)}{m(m+1)(m-1)} a_0$$

put $m=2$, the 1st solution is

$$y_1 = a_0 x^2\left[1+\frac{3}{1}x + \frac{3(4)}{2}x^2 + \cdots\right]$$

$$y_1 = a_0 x^2\left[1+3x+6x^2+10x^3+\cdots\right].$$

Put $m_2=0$, the coefficient become infinite. so put $a_0 =$

$b_0(m-m_2) = b_0(m-0) = mb_0$

$$y = b_0 x^m\left[m + \frac{m(m+1)}{m-1}x + \frac{(m+1)(m+2)}{(m-1)}x^2 + \cdots\right]$$

$$\frac{dy}{dm} = b_0 x^m \log x\left[m + \frac{m(m+1)x}{m-1} + \frac{(m+1)(m+2)}{(m-1)}x^2 + \cdots\right] + b_0 x^m$$

$$\left[1+\frac{m^2-2m-1}{(m-1)^2}x + \frac{m^2-m-5}{(m-1)^2}x^2 + \cdots\right]$$

$$\left(\frac{dy}{dm}\right)_{m=0} = b_0 \log x\left[2x^2-6x^3-12x^4-\cdots\right] + b_0\left[1-x-5x^2-11x^3-\cdots\right]$$

$$= y_2$$

The Complete Solution is

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 a_0 x^2 [1 + 3x + 6x^2 + 10x^3 + \dots] + C_2 b_0 \log x [-2x^2 - 6x^3 + 12x^4 \dots]$$

$$C_2 b_0 [1 - x - 5x^2 - 11x^3 - \dots].$$

$$y = [C_1 a_0 - 2C_2 b_0 \log x] (x^3 + 3x^3 + 6x^4 + \dots) + C_2 b_0 [1 - x - 5x^2 - 11x^3.$$

6. Solve in series $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)y = 0$

Sol:- $x=0$ is singular point

$\because$ Coefficient of $\frac{d^2y}{dx^2} = 0$ at $x=0$

The solution is

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots \quad \text{—— (1)}$$

$$\frac{dy}{dx} = m a_0 x^{m-1} + a_1 (m+1) x^m + (m+2) a_2 x^{m+1} + \dots$$

$$\frac{d^2y}{dx^2} = m(m-1) a_0 x^{m-2} + a_1 m(m+1) x^{m-1} + (m+1)(m+2) a_2 x^m + \dots$$

Sub y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in given equation.

$$x^2 [m(m-1) a_0 x^{m-2} + a_1 m(m+1) x^{m-1} + (m+1)(m+2) a_2 x^m + \dots] +$$

$$x [m a_0 x^{m-1} + a_1 (m+1) x^m + (m+2) a_2 x^{m+1} + \dots] + (x^2 - 4) [a_0 x^m$$

$$- a_1 x^{m+1} + a_2 x^{m+2} + \dots] = 0$$

Comparing Coefficient of least power of x (i.e., x^m) to zero

$$a_0 (m)(m-1) + m a_0 - 4a_0 = 0$$

$$a_0 [m(m-1) + m - 4] = 0$$

$$a_0 [m^2 - \cancel{m} + \cancel{m} - 4] = 0$$

$m^2 = 4 \quad \Rightarrow \quad m = \pm 2$

The roots are distinct and differ by integer

Equating coeff of different powers of x to zero :-

$a_1 m(m+1) + a_1(m+1) - 4a_1 = 0$ [Coefficient of $x^{m+1} = 0$]

$(m+1)(m+1)(a_1) - 4a_1 = 0$

$a_1 = 0$

$(m+1)(m+2)\,a_2 + (m+2)a_2 + a_0 - 4a_2 = 0$

[coefficient of $x^{m+2} = 0$]

$a_2[(m+2)^2 - 4] = -a_0$

$$a_2 = \frac{-a_0}{m(m+4)}$$

Coefficient of x^{m+3} :-

$(m+2)(m+3)a_3 + (m+3)\,a_3 + a_1 - 4a_3 = 0$

$a_3 = 0$

Coefficient of $x^{m+4} = 0$

$(m+3)(m+4)\,a_4 + (m+4)\,a_4 + a_2 - 4a_4 = 0$

$a_4[(m+4)^2 - 4] = -a_2$

$$a_4 = \frac{-a_2}{(m+2)(m+6)} = \frac{a_0}{m(m+2)(m+4)(m+6)}$$

Substituting in eqn (1) we get

$$y = a_0 x^m \left[1 - \frac{x^2}{m(m+4)} + \frac{x^4}{m(m+2)(m+4)(m+6)} \cdots\right]$$

Put $m = 2$

$$y_1 = a_0x^2\left[1 - \frac{x^2}{2(6)} + \frac{x^4}{2(4)(6)(8)} - \frac{x^6}{2(4)(6)^2(8)(10)} + \dots\right]$$

put $m=-2$, the Coefficients become infinite, so sub

$a_0 = b_0(m+2)$ in y.

$$y = b_0x^m\left[(m+2)\left[1 - \frac{x^2}{m(m+4)}\right] + \frac{x^4}{m(m+4)(m+6)} \dots\right]$$

$$\frac{dy}{dm} = b_0x^m\log x\left[(m+2)\left[1 - \frac{x^2}{m(m+4)}\right] + \frac{x^4}{m(m+4)(m+6)} \dots\right] + b_0x^m$$

$$\left[1 - \frac{m+2}{m(m+4)}\left[\frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4}\right]x^2 + \frac{1}{m(m+4)(m+6)}\left[\frac{-1}{m} - \frac{1}{m+4}\right.\right.$$

$$\left.\left.-\frac{1}{m+6}\right]x^4 + \dots\right]$$

$$\left(\frac{dy}{dm}\right)_{m=-2} = b_0x^{-2}\log x\left[\frac{-x^4}{(2)^2(4)} + \frac{x^6}{(2)^3(4)(6)}\right] + b_0x^{-2}\left[1 + \frac{x^2}{4} + \frac{x^4}{(4)(4)^2} \dots\right]$$

The Complete Solution is

$$y = C_1y_1 + C_2y_2$$

$$y = C_1y_1 + C_2\left(\frac{dy}{dm}\right)_{m=-2}$$

$$y = C_1a_0x^2\left[1 - \frac{x^2}{12} + \frac{x^4}{2(4)(6)(8)} + \frac{-x^6}{(2)(4)(6)^2(8)(10)}\right] + C_2b_0x^{-2}\log x\left[\frac{-x^4}{2^2(4)} +\right.$$

$$\left.\frac{x^6}{(2)^3(4)(6)}\right] + C_2b_0x^{-2}\left[1 + \frac{x^2}{(2)^2} + \frac{x^4}{(2)^2(4)^2} + \dots\right].$$

$$y = C_1a_0x^2\left[1 - \frac{x^2}{12} + \frac{x^4}{(2)(4)(6)(8)} - \frac{x^6}{(2)(4)(6)^2(8)(10)}\right] + C_2b_0x^{-2}\left[\log x\right.$$

$$\left.\left\{\frac{-x^4}{(2)^2(4)} + \frac{x^6}{(2)^3(4)(6)}\right\} + \left\{1 + \frac{x^2}{(2)^2} + \frac{x^4}{(2)^2(4)^2} + \dots\right\}\right]$$

7. Solve in series $xy'' + 2y' + xy = 0$.

Sol $x=0$ is singular point.

$\because$ Coefficient $y'' = 0$ at $x=0$

$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots$

$y' = \frac{dy}{dx} = a_0 m . x^{m-1} + a_1(m+1)x^m + a_2x^{m+1}(m+2) + \dots$

$y'' = \frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + a_1 m(m+1)x^{m-1} + (m+1)(m+2)a_2x^m + \dots$

Sub y, y', y'' in given equation.

$x[m(m-1)a_0x^{m-2} + a_1 m(m+1)x^{m-1} + (m+1)(m+2)a_2x^m + \dots] +$

$2[a_0 m x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+2} + \dots] + x[a_0x^m + a_1x^{m+1} + a_2x^{m+2} +$

$a_3x^{m+3} + \dots] = 0.$

Equating coefficient of least power of x to zero:-

Coefficient of $x^{m-1} = 0$

$m(m+1)a_0 + 2ma_0 = 0$

$a_0(m^2 - m + 2m) = 0 \Rightarrow m=0; m=-1$

The roots are distinct and differ by an integer.

Coefficient of $x^m = 0$

$(m+1)(m+2)a_1 = 0.$ $m=-1$ satisfies the condition.

Since $a_1 \neq 0$.

Complete solution is obtained by putting $m=-1$ in y interms of a_0 and a_1.

Equating Coefficient of various powers of x to zero.

$$(m+2)(m+3)a_2 + a_0 = 0$$

$$a_2 = \frac{-a_0}{(m+2)(m+3)} \qquad [\text{Coefficient of } x^{m+1} = 0]$$

$$(m+3)(m+4)\, a_3 + a_1 = 0$$

$$a_3 = \frac{-a_1}{(m+3)(m+4)} \qquad [\text{coefficient of } x^{m+2} = 0]$$

$$(m+4)(m+5)\, a_4 + a_2 = 0 \qquad [\text{coefficient of } x^{m+3} = 0]$$

$$a_4 = \frac{-a_2}{(m+4)(m+5)} = \frac{a_0}{(m+2)(m+3)(m+4)(m+5)}$$

$$(m+5)(m+6)a_5 + a_3 = 0.$$

$$a_5 = \frac{-a_3}{(m+5)(m+6)}$$

$$a_5 = \frac{+a_1}{(m+3)(m+4)(m+5)(m+6)}$$

substituting in y, we get

$$y = x^m \left[a_0 + a_1 x - \frac{a_0 x^2}{(m+2)(m+3)} - \frac{a_1 x^3}{(m+3)(m+4)} + \frac{a_0 x^4}{(m+2)(m+3)(m+4)(m+5)} + \frac{a_1 x^6}{(m+3)(m+4)(m+5)(m+6)} + \dots \right]$$

put $m = -1$

$$y = x^{-1}\left[a_0\left\{1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right\} + a_1\left\{x - \frac{x^3}{6} + \frac{x^5}{120}\right\}\right]$$

Q. prove that $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{(3-x^2)}{x^2}\sin x - \frac{3}{x}\cos x\right\}$.

A. W.K.T $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$ ——— (1)

To get $J_{5/2}(x)$; put $n = 3/2$

$J_{5/2}(x) = \frac{3}{x}\left[J_{3/2}(x)\right] - J_{1/2}(x)$ ——— (2)

find $J_{3/2}(x)$ and $J_{1/2}(x)$,

W.K.T $J_n(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\,\Gamma(n+r+1)}$

put $n = 1/2$

$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\Gamma(3/2)} - \frac{1}{1!\,\Gamma(5/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(7/2)}\left(\frac{x}{2}\right)^4 \dots\right]$

$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\Gamma\frac{1}{2}} - \frac{1}{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^2 + \frac{1}{2.\frac{5}{2}(\frac{3}{2})(\frac{1}{2})\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^4 + \dots\right]$

$= \sqrt{\frac{x}{2\pi}}\left[\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} + \dots\right]$

$= \sqrt{\frac{2}{\pi x}}\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right]$

$= \sqrt{\frac{2}{\pi x}}\sin x$

lly, $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$.

$J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x)$ {from (1) put $n = 1/2$}

$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$

Sub $J_{3/2}(x)$ and $J_{1/2}(x)$ in (2)

$$J_{5/2}(x) = \frac{3}{x}\left[\sqrt{\frac{2}{\pi x}}\left\{\frac{\sin x}{x} - \cos x\right\}\right] - \sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right].$$

9. Prove that $J_n''(x) = \frac{1}{4}\left[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\right]$.

Sol W.K.T.

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \quad \text{--- (1)}$$

diff both sides, we get

$$J_n''(x) = \frac{1}{2}\left[J_{n-1}'(x) - J_{n+1}'(x)\right] \quad \text{--- (2)}$$

Put $n = n-1$ in eqn (1)

$$J_{n-1}'(x) = \frac{1}{2}\left[J_{n-2}(x) - J_n(x)\right] \quad \text{--- (3)}$$

Put $n = n+1$ in eqn (1)

$$J_{n+1}'(x) = \frac{1}{2}\left[J_n(x) - J_{n+2}(x)\right] \quad \text{--- (4)}$$

Sub (3) & (4) in eqn (2)

$$J_n''(x) = \frac{1}{2}\left[\left\{J_{n-2}(x) - J_n(x)\right\}\frac{1}{2} - \frac{1}{2}\left\{J_n(x) - J_{n+2}(x)\right\}\right]$$

$$J_n''(x) = \frac{1}{4}\left[J_{n-2}(x) - J_n(x) - J_n(x) + J_{n+2}(x)\right].$$

$$J_n''(x) = \frac{1}{4}\left[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\right].$$

10 Prove that $\frac{d}{dx}\left[xJ_n(x).J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$

Sol: $\frac{d}{dx}\left[xJ_n(x).J_{n+1}(x)\right] = J_n(x).J_{n+1}(x) + xJ_n(x)J'_{(n+1)}(x) + xJ_n'(x)J_{n+1}$ —— (1)

W.k.T $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$ —— (2)

Put $n = n+1$

$$J'_{n+1}(x) = \frac{n+1}{x}J_{n+1}(x) - J_{n+2}(x) \quad —— (3)$$

W.k.T $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$

Put $n = n+1$

$$J_{n+2}(x) = \frac{2(n+1)}{x}J_{n+1}(x) - J_n(x)$$

Sub in (3)

we get $J'_{n+1}(x) = \frac{n+1}{x}J_{n+1}(x) - \left[\frac{2(n+1)}{x}J_{n+1}(x) - J_n(x)\right]$

$$J'_{n+1}(x) = J_n(x) - \left(\frac{n+1}{x}\right)J_{n+1}(x) \quad —— (4)$$

Sub (2) and (4) in eq (1)

$$\frac{d}{dx}\left[xJ_n(x).J_{n+1}(x)\right] = J_n(x)J_{n+1}(x) + J_n(x)\left[J_n(x) - \left(\frac{n+1}{x}\right)J_{n+1}(x)\right] + x\left[\frac{n}{x}J_n(x) - J_{n+1}(x)\right]J_{n+1}(x)$$

$$= J_n(x)J_{n+1}(x) + x\left[J_n^2(x)\right] - (n+1)J_{n+1}(x)J_n(x) + nJ_n(x)J_{n+1}(x) - x\left[J_{n+1}^2(x)\right]$$

$$= J_n(x)J_{n+1}(x) + xJ_n^2(x) - nJ_{n+1}(x)J_n(x) - J_{n+1}(x)J_n(x) + nJ_{n+1}(x)J_n(x) - xJ_{n+1}^2(x)$$

$$= x\left[J_n^2(x) - J_{n+1}^2(x)\right]$$

$$\therefore \frac{d}{dx}\left[x J_n(x)\, J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$$

Hence proved.

11. Prove that a) $\int J_3(x)dx = C - J_2(x) - \frac{2}{x}J_1(x)$

b) $\int x J_0^2(x)dx = \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right]$

Sol:- a) $\int J_3(x)dx = C - J_2(x) - \frac{2}{x}J_1(x)$

W.K.T $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$

$\Rightarrow \int x^{-n}J_{n+1}(x)dx = -x^{-n}J_n(x)$ —— 1

Now $\int J_3(x)dx = \int x^2 x^{-2} J_3(x)dx + C \Rightarrow$ Constant

$= x^2\int x^{-2}J_3(x)\,dx - \int 2x\left[\int x^{-2}J_3(x)dx\right]dx$

{Put $n=2$ in 1}

$\int J_3(x)dx = x^2\left[-x^{-2}J_2(x)\right] - \int 2x\left[-x^{-2}J_2(x)\right]dx + C.$

$= -J_2(x) + \int \frac{2}{x}J_2(x)\,dx + C.$

$= -J_2(x) + 2\int x^{-1}J_2(x)\,dx + C$ {Put $n=1$ in 1}

$= -J_2(x) - 2x^{-1}J_1(x) + C$

$\int J_3(x)dx = C - J_2(x) - \frac{2}{x}J_1(x)$

Hence proved.

b) $\int x J_0^2(x)\,dx = \int J_0^2(x).\, x\,dx$

$$= J_0^2(x).\frac{x^2}{2} - \int 2J_0(x).J_0'(x).\frac{x^2}{2}\,dx$$

$$= \frac{x^2 J_0^2(x)}{2} - \int x^2 J_0(x)\left[-J_1(x)\right]dx$$

$$\left\{ \begin{array}{l} \because J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x) \\ n=0 \Rightarrow J_0'(x) = -J_1(x) \end{array} \right\}$$

$$= \frac{x^2 J_0^2(x)}{2} + \int x J_1(x).\frac{d}{dx}\left[x J_1(x)\right]dx$$

$$\left\{ \because \frac{d}{dx}\left[x J_1(x)\right] = x J_0(x) \right\}$$

$$\Rightarrow \int x J_0^2(x)\,dx = \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right]$$

Hence proved.

12. Show that a) $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$, n integer.

b) $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos x(\cos\phi)\,d\phi$.

Sol: a) $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$, n is integer.

W.K.T $e^{\frac{1}{2}x(t-1/t)} = J_0(x) + tJ_1(x) + t^2J_2(x) + \dots + t^{-1}J_1(x) + t^{-2}J_{-2}(x) + \dots$

$$\{\because J_n(x) = (-1)^n J_n(x)\}$$

$$e^{\frac{1}{2}x(t-1/t)} = J_0 + J_1(t - 1/t) + J_2(t^2 + 1/t^2) + J_3(t^3 + 1/t^3) + \dots \longrightarrow (1)$$

Put $t = \cos\theta + i\sin\theta$.

So, $1/t = \cos\theta - i\sin\theta$

Sub in ①, we get

$$e^{ix\sin\theta} = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta + \cdots] + 2i[J_1\sin\theta + J_3\sin 3\theta + \cdots]$$

$$\therefore e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta).$$

Equating real and imaginary parts, we get

$$\cos(x\sin\theta) = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta + \cdots] \quad -②$$

$$\sin(x\sin\theta) = 2[J_1\sin\theta + J_3\sin 3\theta + \cdots] \quad -③$$

Now multiply both sides of ② by $\cos n\theta$ and both sides of ③ by $\sin n\theta$ and integrating each from 0 to π, we get

$$\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos n\theta \, d\theta = \begin{cases} J_n(x) & n \text{ even (or) zero} \\ 0 & n \text{ odd.} \end{cases}$$

$$\frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin n\theta \, d\theta = \begin{cases} 0 & n \text{ even} \\ J_n(x) & n \text{ odd.} \end{cases}$$

If n is an integer

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta] \, d\theta.$$

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta) \, d\theta$$

b. $$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos x(\cos\phi) \, d\phi$$

W.K.T

$$\cos(x\sin\theta) = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta + \cdots]$$

Put $\theta = \frac{1}{2}\pi - \phi$, we get

$$\cos(x\cos\phi) = J_0 + 2J_2\cos(\pi - 2\phi) + 2J_4\cos(2\pi - 4\phi) + \cdots$$

$$= J_0 - 2J_2\cos 2\phi + 2J_4\cos 4\phi \cdots$$

Integrating both the sides w.r.t ϕ from 0 to π.

we get

$$\int_0^\pi \cos x(\cos\phi)\,d\phi = \int_0^\pi [J_0(x) - 2J_2(x)\cos 2\phi + 2J_4\cos 4\phi \ldots]\,d\phi$$

$$= \left[J_0(x)\phi - 2J_2(x)\cdot\frac{1}{2}\sin 2\phi + 2J_4(x)\cdot\frac{1}{4}\sin 4\phi - \ldots\right]_0^\pi$$

$$= J_0(x)\cdot\pi \quad \{\because \sin n\pi = 0\}$$

$$\Rightarrow J_0(x) = \frac{1}{\pi}\int_0^\pi \cos x(\cos\phi)\,d\phi.$$

Hence proved.

13. Show that $J_0^2 + 2J_1^2 + 2J_2^2 + \ldots = 1$

Sol: W.K.T

$$\cos(x\sin\theta) = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta + \ldots] \quad —① $$

$$\sin(x\sin\theta) = 2[J_1\sin\theta + J_3\sin 3\theta + \ldots] \quad —②$$

squaring ① and ② and integrating from 0 to π we get

$$\int_0^\pi \cos^2(x\sin\theta)\,d\theta = \int_0^\pi [J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \ldots]^2\,d\theta.$$

$$\Rightarrow \int_0^\pi \cos^2(x\sin\theta)\,d\theta = \int_0^\pi [J_0^2 + 4J_2^2\cos^2 2\theta + 4J_4^2\cos^2 4\theta + \ldots]\,d\theta$$

$$\int_0^\pi \cos^2(x\sin\theta)\,d\theta = J_0^2(\pi) + 4J_2^2(\pi/2) + 4J_4^2(\pi/2) + \ldots$$

$$\Rightarrow \int_0^\pi \cos^2(x\sin\theta)\,d\theta = [J_0(x)]^2\pi + 2[J_2(x)^2]\pi + 2[J_4(x)]^2(\pi) + \ldots \quad —③$$

from ②

$$\int_0^\pi \sin^2(x\sin\theta)\,d\theta = \int_0^\pi 4[J_1\sin\theta + J_3\sin 3\theta + \ldots]^2\,d\theta$$

$$\left\{\because \int_0^\pi \sin m\theta\cdot\sin n\theta\,d\theta = 0\right\}$$

$$\int_0^{\pi} \sin^2(x\sin\theta)d\theta = 4\int_0^{\pi}[J_1^2\sin^2\theta + J_3^2\sin^2 3\theta + \dots]d\theta$$

$$\left\{\because \int_0^{\pi} \sin^2 n\theta\, d\theta = \pi/2\right\}.$$

$$\int_0^{\pi} \sin^2(x\sin\theta)d\theta = 4[J_1^2(x)\pi/2 + J_3^2(x)\pi/2 + \dots].$$

$$\int_0^{\pi} \sin^2(x\sin\theta)\,d\theta = 2[J_1(x)]^2\pi + 2[J_3(x)]^2\pi + \dots \quad -(4)$$

$$(3)+(4) \Rightarrow \int_0^{\pi} \cos^2(x\sin\theta)d\theta + \sin^2(x\sin\theta)d\theta = \pi[J_0^2 + 2J_1^2 + 2J_2^2 + \dots]$$

$\Rightarrow J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \dots = 1.$ hence proved.

14. Reduce the differential equation $x\frac{d^2y}{dx^2} + c\frac{dy}{dx} + k^2x^{2n}y = 0$ to bessel form by putting $x = t^m$.

Sol $$x\frac{d^2y}{dx^2} + c\frac{dy}{dx} + k^2x^q dy = 0 \quad -(1).$$

$$x = t^m \Rightarrow t = x^{1/m}.$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{t^{1-m}}{m}\cdot\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{m}t^{1-m}\frac{dy}{dt}\right)\cdot\frac{1}{m}t^{1-m}.$$

$$= \frac{1}{m^2}t^{2-2m}\frac{d^2y}{dt^2} + \frac{1-m}{m^2}t^{1-2m}\frac{dy}{dt}$$

Sub $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and y in eq (1)

$$\frac{1}{m^2}t^{2-m}\frac{d^2y}{dt^2} + \frac{1-m+cm}{m^2}t^{1-m}\frac{dy}{dt} + k^2t^{mq}y = 0.$$

multiply by m^2/t^{1-m}, we get

$$t^2\frac{d^2y}{dt^2} + (1-m-cm)\frac{dy}{dt} + (km)^2 t^{mq+m-1}y = 0 \quad -(2)$$

To reduce the above eq, Put $ma + m - 1 = 1$

i.e., $m = 2/(a+1)$;

let $a = 1 - m + cm = (a+2c-1)/a+1$

Now eq ② becomes.

$$t \frac{d^2y}{dt^2} + a \frac{dy}{dt} + (km)^2\, t y = 0$$

The solution to this equation is

$$y = x^{n/m}\left[c_1 J_n(k(m) x^{1/m}) + c_2 J_n(km\, x^{1/m})\right].$$ n is fraction

$$y = x^{n/m}\left[c_1 J_n(km x^{1/m}) + c_2 Y_n(km\, x^{1/m})\right],$$ n is integer

$$n = \frac{1-a}{2} = \frac{1-c}{1+a} \quad \& \quad m = \frac{2}{1+a}.$$

15 Solve the differential equation.

a) $y'' + y'/x + (8 - \frac{1}{x^2})\, y = 0$

b) $4y'' + 9xy = 0$.

Sol: a) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (8x^2 - 1)\, y = 0$

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (k^2x^2 - n^2)\, y = 0$$ (general form).

On Comparing ; $n = 1$; $k = 2\sqrt{2}$

The solution of given Equation is

$$y = C_1 J_n(kx) + C_2 Y_n(kx)$$

$$y = C_1 J_1(2\sqrt{2}x) + C_2 Y_1(2\sqrt{2})x$$

b) $4y'' + 9xy = 0$

$$x\frac{d^2y}{dx^2} + \frac{9}{4}x^2 y = 0$$

Comparing the above equation with.

$$t\frac{d^2y}{dt^2} + a\frac{dy}{dt} + (km)^2 t y = 0$$

we get $c=0$; $k = 3/2$, $\alpha = 2$

$$n = \frac{1-c}{1+\alpha} = \frac{1}{3} ; \quad m = \frac{2}{1+\alpha} = \frac{2}{3} ; \quad \frac{n}{m} = 1/2$$

The solution to equation is

$$y = x^{n/m}\left[c_1 J_n(kmx^{1/m}) + c_2 Y_n(kmx^{1/m})\right)$$

$$y = \sqrt{x}\left[c_1 J_{1/3}(x^{3/2}) + c_2 Y_{-n}(x^{3/2})\right].$$

16 Solve the differential equation $xy'' + y' + 1/4\, y = 0$.

Sol: $xy'' + y' + 1/4\, y = 0$.

Multiply by x; $\quad x^2y'' + xy' + 1/4\, xy = 0$.

Comparing with $x\frac{d^2y}{dx^2} + c\frac{dy}{dx} + k^2 x^\alpha y = 0$

we get ; $c=1$; $k = 1/2$; $\alpha = 0$.

$$m = \frac{2}{1+\alpha} = 2 ; \quad n = \frac{1-c}{1+\alpha} = 0 ; \quad \frac{n}{m} = 0$$

The solution to the equation is

$$y = x^{n/m}\left[c_1 J_n(kmx^{1/m}) + c_2 Y_n(kmx^{1/m})\right]$$

$$y = c_1 J_0(\sqrt{x}) + c_2 Y_0(\sqrt{x}).$$

17 Explain the orthogonality of Bessel function.

Sol: We know that the solution of the equation.

$$x^2 u'' + xu' + (\alpha^2 x^2 - n^2) u = 0 \quad \text{—} ①$$

$$x^2 v'' + xv' + (\beta^2 x^2 - n^2) v = 0 \quad \text{—} ②$$

$u = J_n(\alpha x)$ and $v = J_n(\beta x)$ respectively

Multiplying ① by $\frac{v}{x}$ and ② by $\frac{u}{x}$ and substracting ① and ② we get

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2) xuv = 0$$

$$\frac{d}{dx}\left[x(u'v - uv')\right] = (\beta^2 - \alpha^2) x uv.$$

Now integrating both sides from 0 to 1

$$(\beta^2 - \alpha^2) \int x uv \, dx = \left[x(u'v - uv')\right]_0^1$$

$$= (u'v - uv')_{x=1}$$

$u = J_n(\alpha x)$.

$$u' = \frac{d}{dx}\left[J_n(\alpha x)\right] = \frac{d}{d(\alpha x)}\left[J_n(\alpha x)\right] \cdot \frac{d(\alpha x)}{dx} = \alpha J_n'(\alpha x)$$

If α, β are distinct roots of $J_n(x) = 0$ then $J_n(\alpha) = 0$, $J_n(\beta) = 0$; then $\int_0^1 x J_n(\alpha x) J_n(\beta x) \, dx = 0$

This is known as the orthogonality relation of Bessel function.

18. If $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are the positive roots of $J_0(x)=0$.

show that $\frac{1}{2} = \sum_{n=1}^{\infty} \left[J_0(\alpha_n x) / \alpha_n J_1(\alpha_n) \right]$

Sol: If $f(x) = C_1 J_n(\alpha_1 x) + C_2 J_n(\alpha_2 x) + \ldots + C_r J_n(\alpha_n x) + \ldots$

$$C_r = \frac{2}{\alpha^2 J_{n+1}^2(a\alpha_r)} \int_0^a x f(x) J_n(\alpha_r x)\, dx$$

Taking $f(x)=1$; $a=1$ and $n=0$, we get

$$C_r = \frac{2}{J_1^2(\alpha_r)} \int_0^1 x J_0(\alpha_r x)\, dx$$

$$= \frac{2}{J_1^2(\alpha_r)} \left| \frac{x J_1(\alpha_r x)}{\alpha_r} \right|_0^1$$

$$= \frac{2}{\alpha_r J_1(\alpha_r)}$$

$$1 = \sum_{r=1}^{\infty} \frac{2}{\alpha_r J_1(\alpha_r)} J_0(\alpha_r x)$$

$$\Rightarrow \frac{1}{2} = \sum_{n=1}^{\infty} \frac{J_0(\alpha_n x)}{\alpha_n J_1(\alpha_n)}$$

Hence proved.

19 Expand $f(x) = x^2$ in the interval $0 < x < 2$ in terms of $J_2(\alpha_n x)$, where α_n are determined by $J_2(2\alpha_n) = 0$

Sol: Let the fourier - bessel expansion of $f(x)$ be

$$x^2 = \sum_{n=1}^{\infty} C_n J_2(\alpha_n x)$$

Multiplying both sides by $x J_2(\alpha_n x)$ and integrating w.r.t x from 0 to 2, we get

$$\int_0^2 x^3 J_2(\alpha_n x)\,dx = C_n \int_0^2 x J_2^2(\alpha_n x)\,dx$$

$$= C_n \frac{(2)^2}{2} J_3^2(2\alpha_n).$$

$$\left[\frac{x^3 J_3(\alpha_n x)}{\alpha_n}\right]^2 = 2C_n J_3^2(2\alpha_n)$$

$$C_n = \frac{4}{\alpha_n J_3(2\alpha_n)}.$$

$$\therefore x^2 = 4\sum_{n=1}^{\infty} \frac{J_2(\alpha_n x)}{\alpha_n J_3(2\alpha_n)}.$$

SPECIAL FUNCTION –II

UNIT IV

1) Write the Legendre's equation. (K1)
2) Write the Legendre's polynomial of order n (K2)
3) Define the Legendre function of the second kind (K1)
4) Write the Rodrigues's formula (K1)
5) Write the generating function of Legendre polynomials (K2)
6) Write the orthogonality property of Legendre polynomials (K1)
7) Write the Fourier- Legendre expansion of f(x) from x=-1 to 1 (K1)
8) Define Legendre's polynomials (K2)
9) Define Hermit's polynomials (K1)
10) Define Chebyshev polynomials (K2)
11) Write the strum- Liouville equation (K1)
12) Define orthonormal on $a \leq x \leq b$ (K1)
13) Define orthogonality of Legendre polynomials (K2)
14) Define orthogonality of Bessel function (K1)
15) Write the polynomial $2x^2+x+3$ in terms of Legendre polynomials (K1)
16) If $P_n(x)$ be the Legendre polynomial then write the polynomial equation to $P_n'(-x)$ (K1)
17) What is λ when $P_5(x) = \lambda(63x^5-70x^3+15x)$ a Legendre polynomial? (K2)
18) Find $\int_{-1}^{1}(1+x)p_n(x)dx$ (n>1) (K1)
19) Write the singular points of the differential equation $x^3(x-1)y''+2(x-1)y'+y=0$ (K2)

20) Find the value of the integral $\int_{-1}^{1}x^3 p_3(x)dx$ where $P_3(x)$ is a Legendre polynomial of degree 3 (K1)

unit - IV.

Assignment -7.

Part -A:

① Write legendre's equation?

Sol: $$(1-x^2)\frac{d^2y}{dx^2} - (2x)\frac{dy}{dx} + n(n+1)y = 0$$

where $n \rightarrow$ is a real number.

2. Write the legendre's polynomial of order n.

A. $$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}.$$

$M = n/2$ (or) $(n-1)/2$, whichever is an integer.

3. Define the legendre function of second kind.

A. The infinite series solution with a_0 (or) a_1 properly chosen is called legendre function of second kind

It is denoted by $Q_n(x)$

4. Write the Rodrigue's's formula

A. $$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n$$

5. Write the generating function of lengendre polynomials.

A. $$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$$

$P_n(x)$ is the Coefficient of t^n in the expansion of $(1-2x+t^2)^{-1/2}$. It is known as the generating function of longendre polynomials.

6. Write the orthogonality property of legendre polynomial

A. $$\int_{-1}^{1} P_m(x) P_n(x)\,dx = 0 \qquad m \neq n$$

This is known as the orthogonality property of legendre polynomial.

7. Write the fourier legendre expansion of $f(x)$ from $x = -1$ to 1

A. If $f(x)$ be a function defined from $x = -1$ to 1 then

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x)$$

$$\int_{-1}^{1} f(x) P_n(x)\,dx = C_n \int_{-1}^{1} P_n^2(x)\,dx$$

$$= \frac{2C_n}{2n+1}$$

$$C_n = (n + 1/2) \int_{-1}^{1} f(x) P_n(x)\,dx$$

8. Define legendre's polynomials.

A. Legendre's polynomials are a system of complete and orthogonal polynomials, are with a vast number of mathematical properties and numerous application.

9. Define hermite polynomials?

A. These are the solutions of hermite differential equation $y'' - 2xy' + 2ny = 0$. These polynomials $H_n(x)$ are given by Rodrigues formula.

$$H_n(x) = (-1)^n e^{x^2} d^{2n} (e^{-x^2})$$

10. Define chebyshev polynomials?

A. Chebyshev polynomials of degree $n \geq 0$ is defined as

$T_n(x) = \cos(n \arccos x)$, $x \in (-1, 1)$

$T_n(x) = \cos n\theta$.

$x = \cos\theta$, $\theta \in [0, \pi]$.

11. Write the strum-liouvills equation.

A. $$[r(x) y']' + [\lambda p(x) + q(x) y] = 0$$

This is known as strum-liouvills equation

12. Define orthogonal on $a \leq x \leq b$.

A. The function which are orthogonal on $a \leq x \leq b$ and have norm equal to 1, are called orthogonal on this interval $\{a \leq x \leq b\}$

13. Define orthogonality of legendre polynomial.

A. The legendre's polynomial $P_m(x)$ & $P_n(x)$ are said to be orthogonal in the interval $-1 \leq x \leq 1$ if

$$\int_{-1}^{1} P_m(x) \cdot P_n(x) \cdot dx = 0 \quad \text{for } m \neq n$$

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \frac{2}{2n+1} \quad \text{for } m = n$$

14. Define orthogonality of Bessel's function.

If α and β are distrinct roots of $J_n(x) = 0$ the $J_n(\alpha) = J_n(\beta) = 0$

$$\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = 0.$$

If $\alpha = \beta$, then $\int_0^1 x J_n^2(\alpha x) = \frac{1}{2}[J_{n+1}(\alpha)]^2$.

This is known as orthogonality of Bessel's function.

15 Write the polynomial $2x^2+x+3$ in terms of legendre polynomial.

A WKT the legendre polynomials are

$P_0(x) = 1$; $P_1(x) = x$; $P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$; $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$. . . .

from $P_2(x) = \frac{1}{2}(3x^2-1) \Rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$

$P_1(x) = x$

$f(x) = 2x^2+x+3 = 2\left[\frac{2}{3}P_2(x)+\frac{1}{3}\right] + P_1(x) + 3P_0x.$

$= \frac{4}{3}P_2(x) + P_1(x) + \frac{11}{3}P_0(x)$

$= \frac{1}{3}\left[4P_2(x) + 3P_1(x) + 11P_0(x)\right].$

16 If $P_n(x)$ be the legendre's polynomial then write the polynominal equation to $P_n'(-x)$.

A W.K.T $\sum_{n=0}^{\infty} t^n P_n(x) = [1-2xt+t^2]^{-1/2}$

Put $x=-x$; $\sum_{n=0}^{\infty} t^n P_n(-x) = (1+2xt+t^2)^{-1/2}$ —— ①

Put $t=-t$; $\sum_{n=0}^{\infty} t^n P_n(x) = (1+2xt+t^2)^{-1/2}$ —— ②

from ① & ② $\sum_{n=0}^{\infty} t^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n t^n P_n(x)$

Comparing coefficient of t^n we get, $P_n(-x) = (-1)^n P_n(x)$ —— ③

$(-1)P_n'(-x) = (-1)^n P_n(x)$

$P_n'(-x) = (-1)^{n+1} P_n(x).$

17. What is d when $P_5(x) = d\,[63x^5 - 70x^3 + 15x]$ a legendre polynomial

Sol: W.K.T $P_5(x) = \frac{1}{8}[63x^5 - 70x^3 + 15x]$

Comparing with given equation.

we know that $d = 1/8$

18. Find $\int_{-1}^{1}(1+x)p_n(x)\,dx \quad (n>1)$

A W.K.T $\int_{-1}^{1} f(x)\,p_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} f^n(x)\,(1-x^2)^n\,dx$

here $f^n(x) = \frac{d^n}{dx^n}[f(x)] = \frac{d^n}{dx^n}[1+x] = 0.$

$\Rightarrow \int_{-1}^{1}(1+x)\,p_n(x)\,dx = 0$

19. Write the singular points of the differentiation equation

$x^3(x-1)y'' + 2(x-1)y + y = 0.$

A For singular points the Coefficient of y'' is 0.

$x^3[x-1] = 0$

$x=0 \;;\; x-1=0.$

$x=1$

$\therefore$ The singular points are 0,1.

20 Find the value of the integral $\int_{-1}^{1} x^3 p_3(x)\,dx$ where $P_3(x)$ is a legendre polynomial of degree 3

Sol. W.K.T $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$\int_{-1}^{1}[P_3(x)]x^3\,dx = \frac{1}{2}\int_{-1}^{1}(5x^6 - 3x^4)\,dx$

$= \frac{1}{2}\left[5x^7/7 - \frac{3x^5}{5}\right]_{-1}^{1}$

$= \frac{1}{2}\left[\frac{5}{7} - \frac{3}{5} + \frac{5}{7} - \frac{3}{5}\right]$

$= \frac{1}{\cancel{2}}\left[\cancel{2}\left(5/7 - 3/5\right)\right]$

$= \cancel{PA+B} \quad 5/7 - 3/5 = 4/35$

Unit – IV

Special function-2

1 Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ interms of legendre polynomials. (K3)

2 Show that for any function f(x) , $\int_{-1}^{1} f(x)\, p_n(x)\, dx = \frac{1}{2^n n!}\int_{-1}^{1}(1 + x^2)^n f^n(x)\, dx$. (K4)

3 Show that $P_n(x) = (n+1)P_{n-1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$. (K3)

4 Show that $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$. (K3)

5 Show that $(2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$. (K4)

6 Show that $P_n'(x) = x P_{n-1}'(x) + n P_{n+1}(x)$ (K3)

7 Prove that $(1-x^2) P_n'(x) = n[P_{n-1}(x) - x P_n(x)]$ (K5)

8 Prove that $(2n+1)(1- x^2) P_n'(x) = n(n+1)[P_{n-1}(x) - P_n(x)]$. (K4)

1 Discuss the orthogonality of legendre polynomials. (K4)

2 Show that $\int_{-1}^{1} x\, P_n(x)\, P_{n-1}(x)\, dx = \frac{2n}{4n^2-1}$. (K5)

3 Show that $\int_{-1}^{1} x^2 P_{n-1}P_{n+1}dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$. (K6)

4 Prove that $\int_{-1}^{1} (1-x^2)P_m'(x)P_n'(x)dx = 0, m \neq n$. (K4)

5 Prove that $\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)dx = \frac{2n(n+1)}{2n+1}$, $m = n$. (K5)

6 If $f(x) = 0, -1 < x \leq 0$

$= x, 0 < x < 1,$

7 Show that $f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + ...$ (K5)

8 Prove that $(2n+1)(1- x^2) P_n'(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]$. (K4)

9. Show that $\int_{0}^{\infty} e^{-x} L_m(x) L_n(x)\, dx = 0, m \neq n$. (K4)

10. Prove that $H_n(x) = (-1)^n e^{x^2} \frac{d^{2n}}{dx^n}\left(e^{-x^2}\right)$ (K3)

11. For that strum – liouville problem y''+λ y = 0, y(0) = 0, y(l) = 0. Find the eigen function and show that they are orthogonal. (K4)

TEST 1 - ECE -MULTIVARIABLE CALCULUS

2 - marks

1. Find $\frac{du}{dx}$ if $u = x^2 y$, where $x^2 + xy + y^2 = 1$. (K2)
2. Write the Taylor's series expansion of $f(x, y)$ about (a, b). (K1)
3. Define a saddle point. (K2)
4. If $u = x^2$ and $v = y^2$, find $\frac{\partial(u,v)}{\partial(x,y)}$. (K1)
5. State Euler's theorem (K2)
6. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ if $u = x^2y - \sin(xy)$ (K1)

16 = marks

1. Find the extreme values of the function $f(x, y) = x^3 + y^3 - 12x - 3y + 20$. (K4)
2. Find the dimensions of the rectangular box without top of maximum capacity with surface area 432 square metre. (K3)
3. Find the extreme values of the function, $f(x, y) = x^3y^2 (1 - x - y)$. (K5)

Test - I

11189C106
K. Gayatri Supraja

2-marks

1. Find $\frac{dy}{dx}$ if $u = x^2 y$, where $x^2 + xy + y^2 = 1$

Sol:- $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$

$\frac{\partial u}{\partial x} = 2xy$, $\frac{\partial u}{\partial y} = x^2$ $\frac{dy}{dx} = 2x + y + 2y\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-2x - y}{2y}$$

$$= 2xy + x^2\left(\frac{-2x - y}{2y}\right)$$

$$= \frac{4xy^2 - 2x^3 - x^2 y}{2y}$$

2. write the Taylor's series expansion of $f(x, y)$ about (a, b), $f(x, y)$ =

$f(a, b) + \frac{1}{1!}\left[x f_x(a,b) + y f_y(a, b)\right] + \frac{1}{2!}\left(x^2 f_{xx}(a,b) + 2xy\right.$

$f_{xy}(a,b) + y^2 f_{yy}(a,b) + \frac{1}{3!}\left[x^3 f_{xxx}(a,b) + 3x^2 y\, f_{xxy} + 3xy^2 f_{xyy}(a,b) + y^3 f_{yyy}(a,b)\right]$

3. Define Saddle point

Ans. A value of function of 2 variables which is a max w.r.to one a minimum w.r.to other it is called saddlepoint. $rt - s^2 < 0$ and $r > 0, r < 0$, then it is called saddlepoint.

4. If $u = x^2$ and $v = y^2$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy$

5. State Euler's theorem.

Ans :- If 'u' is a homogeneous function of degree, n in x & y then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot u \quad \text{where } n = \text{degree}.$$

6. Find $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$ if $u = x^2 y - \sin(xy)$

Sol:- $\frac{\partial u}{\partial x} = 2xy - xy\cos xy - \sin xy$

$\frac{\partial u}{\partial y} = x^2 \cdot 1 = x\cos xy \cdot x$

$= x^2 - x^2 \cos xy.$

16 Marks

1. Find the values of the function $f(x,y)=x^3+y^3-12x-3y+20$

Sol: $f(x,y) = x^3+y^3-12x-3y+20$ —— ①

$$\frac{\partial f}{\partial x} = 3x^2-12=0 \quad \text{—— ②}$$

$$\frac{\partial f}{\partial y} = 12y^2-12=0 \quad \text{—— ③}$$

solve ② & ③

$3x^2-12=0$

$12y^2-12=0$

$3x^2-12y^2=0$

$x=2y$

Sub $x=2y$ in ③

$12y^2-12=0$

$12y^2-12 \Rightarrow y^2=1 \Rightarrow y=\pm 1$

$y=1 \Rightarrow x=2$

$x=-1 \Rightarrow x=-2$

paired values :- $(2,1)$; $(2,-1)$; $(-2,1)$; $(-2,-1)$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$s = \frac{\partial f}{\partial x \partial y} = 0$$

$$t = \frac{\partial^2 t}{\partial y^2} = 6y$$

paired values	r	t	rt	s	s^2	$rt-s^2$	result	Value
$(2,1)$	12	6	72	0	0	72	$rt-s^2>0$, $r>0$	minimum value
$(2,-1)$	12	-6	-72	0	0	-72	$rt-s^2<0$, $r>0$	
$(-2,1)$	-12	6	-72	0	0	-72	$rt-s^2<0$, $r<0$	
$(-2,-1)$	-12	-6	72	0	0	72	$rt-s^2>0$, $r<0$	Maximum Value

$$f(2,1) = x^3+y^3-12x-3y+20$$
$$= 8+1-24-3+20 = 2 \ [\text{minimum value}]$$

$$f(-2,-1) = -8-1+24+3+20$$
$$= 38 \ [\text{maximum value}]$$

2. Find the dimensions of a rectangular box without top of max capacity with surface area 432 sq.m.

Sol: Given, let x, y, z be the length, breath and height of the box.

Surface are $= 432$

$$S = xy + 2yz + 2zx = 432$$
$$V = xyz = f$$

Let $y = f + \lambda\phi \Rightarrow g = xyz + \lambda(xy + 2yz + 2zx - 432)$

$$g_x = yz + \lambda(y+2z) = 0 \quad \text{--- (1)}$$
$$g_y = xz + \lambda(x+2z) = 0 \quad \text{--- (2)}$$
$$g_z = xy + \lambda(2y+2x) = 0 \quad \text{--- (3)}$$

$-\lambda = \dfrac{yz}{y+2z}$ — (4) $-\lambda = \dfrac{xz}{x+2z}$ — (5) $-\lambda = \dfrac{xy}{2y+2x}$ — (6)

By solving (4) & (5), we get

$-\lambda = \dfrac{yz}{y+2z}$

$-\lambda = \dfrac{xz}{x+2z}$

$x = y$

sub $x = y$ in (6)

$-\lambda = \dfrac{x}{4}$

$x = -4\lambda$

Similarly, $y = -4\lambda$

$z = -2\lambda$

sub $x, y\ z$ in (5)

$S = xy + 2yz + 2zx = 16\lambda^2 + 16\lambda^2 + 16\lambda^2 = 432$

$48\lambda^2 = 432$

$\lambda^2 = \dfrac{432}{48} \Rightarrow \lambda^2 = 9 \Rightarrow \lambda = \pm 3 = 3$

$\therefore x = 12,\ y = 12,\ z = 6.$

3. Find the extreme value of the function

$f(x,y) = x^3 y^2 (1 - x - y) = x^3 y^2 - x^4 y^2 - x^3 y^3$

Sol:- $f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

$\dfrac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2 = 0$$

Then we have,

$x^2y^2(x - 4\cancel{x} - 3y) = 0 \quad x = 0\ ;\ y = 0\ ;\ 4x + 3y = 3$ — ①

$x^2y\,[2 - 2x - 3y] = 0 \quad x = 0\ ;\ y = 0\ ;\ 2x + 3y = 2$ — ②

Solve ① & ②

$$4x + 3y = 3$$
$$2x + 3y = 2$$
$$(- \quad - \quad -)$$
$$2x = 1$$
$$x = 1/2$$

Sub $x = 1/2$ in ②

$$2\left(\tfrac{1}{2}\right) + 3y = 2$$
$$3y = 1$$
$$y = \frac{1}{3}$$

The stationary points are $(0,0), (0, 1/3), (1/2, 0)\ (1/2, 1/3)$

$$r = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$

pair of values	r	t	rt	s	s^2	$rt - s^2$	result	value
$(0,0)$	0	0	0	0	0	0	fails	fails
$(0, 1/3)$	0	0	0	0	0	0	fails	fails
$(1/2\ 0)$	0	0	0	0	0	0	fails	fails
$(1/2, 1/3) =$	$-1/9$	$-1/8$	$1/72$	$1/12$	$1/144$	$1/144$	$rt - s^2 > 0$ $r < 0$ max	0.0066

TEST 2- ECE - VECTOR CALCULUS

2- marks

1. Find the divergence of the vector point function $xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ (K2)
2. State Green's theorem. (K1)
3. If $\phi(x,y,z) = x^2y + y^2x + z^2$, then find $\nabla\phi$ at the point (1,1,1). (K2)
4. Find the directional derivative of $4x^2z + xy^2$ at the point (1,-1,2) in the direction of the vector $2\bar{i} - \bar{j} + 3\bar{k}$ (K2)
5. Prove that div $\bar{r}$ = 3 and curl $\bar{r}$ = 0. (K2)
6. Define Solenoidal. (K1)

16- marks

1. Show that $\nabla^2(r^n \bar{r}) = n(n+3)r^{n-2}\bar{r}$ with usual notations. (K4)
2. If $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla^2 r^n = n(n+1)r^{n-2}$ and hence deduce $\nabla\left(\frac{1}{r}\right)$. (K4)

3.If $\vec{F} = 3xyz^2\vec{i} + 2xy^3\vec{j} - x^2yz\vec{k}$ & f =3x²-yz find i) $\nabla.\nabla f$ .ii) div (curl) $\vec{F}$ at (1,-1, 1) (K5)

3.

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K. Gayatri Supraja

Part - I (2 marks)

1> Find the divergence of the vector point $xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

Sol:- $\text{div}\,\vec{F} = \nabla \cdot \vec{F}$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

2> State green's theorem.

Sol:- If c is a sample closed curve enclosed in a region R in the xy plane and $p(x,y)$ $Q(x,y)$ and its first order partial derivatives are continuous in R.

then $\int_c (Pdx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$

where c is described in anti clockwise direction.

3> If $\phi(x,y,z) = x^2y + y^2x + z^2$, then find $\nabla\phi$ at point $(1,1,1)$.

Sol:- $\phi(x,y,z) = x^2y + y^2x + z^2$

$$\nabla\phi = 2xy\,\vec{i} + 2yx\,\vec{j} + 2z\vec{k}$$

$$\nabla\phi_{(1,1,1)} = 2\vec{i} + 2\vec{j} + 2\vec{k}$$

$$= 2[\vec{i} + \vec{j} + \vec{k}].$$

4) Find the directional derivative of $4x^2z + xy^2$ at the point $(1,-1,2)$ in the direction of $2\vec{i} - \vec{j} + 3\vec{k}$

Sol:- $D \cdot D = \dfrac{\nabla\phi}{|\nabla\phi|}$ $\Rightarrow$ $\phi = 4x^2z + xy^2$

$\nabla\phi = \vec{i}(8xz + y^2) + \vec{j}(2yx) + \vec{k}(4x^2)$

$\nabla\phi_{(1,-1,2)} = 17\vec{i} - 2\vec{j} + 4\vec{k}$

$$D \cdot D = \frac{17\vec{i} - 2\vec{j} + 4\vec{k}}{\sqrt{2^2 + 1^2 + 3^2}} = \frac{17\vec{i} - 2\vec{j} + 4\vec{k}}{\sqrt{14}}$$

5) Prove that div $\vec{r} = 3$ and Curl $= 0$

Sol: $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x)\vec{i} \cdot \vec{i} + \frac{\partial}{\partial y}(y)\vec{j} \cdot \vec{j} + \frac{\partial}{\partial z}(z)\vec{k} \cdot \vec{k}$$

$$= 1 + 1 + 1 = 3 = \nabla\vec{r} = \text{div}.\vec{r}$$

$$\text{Curl}.\vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

6) Define solenoidal.

Ans If $\vec{F}$ vector is a vector such that $\nabla \vec{F} = 0$ at all points in a given region then it is said to be a solenoidal vector in that region $\nabla \cdot \vec{F} = 0$.

Part - 2 (16 marks)

1) Show that $\nabla^2(r^n \vec{r}) = n(n+3) r^{n-2} \vec{r}$ with usual notations.

Sol:- W.K.T $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$$= \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{n/2} x\right] + \frac{\partial}{\partial y}\left[(x^2+y^2+z^2)^{n/2} y\right]$$

$$+ \frac{\partial}{\partial z}\left[(x^2+y^2+z^2)^{n/2} z\right]$$

$$= nx^2 r^{n-2} + r^n + n\cdot y^2 r^{n-2} + r^n + n\cdot z^2 r^{n-2} + r^n$$

$$= n\cdot r^{n-2}\cdot r^3 + 3r^n = (n+3) r^n = (n+3)(x^2+y^2+z^2)^{n/2}$$

$$\nabla(r^n\cdot\vec{r}) = (n+3)\left[\vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2}\right.$$

$$\left. + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}\right].$$

$$= n(n+3)\left[\vec{i}\, x\cdot r^{n-2} + \vec{j}\cdot y\cdot r^{n-2} + \vec{k}\cdot z\cdot r^{n-2}\right]$$

$$= n(n+3)\, r^{n-2}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= n(n+3)\, r^{n-2}\vec{r}.$$

2) If $\vec{r}$ is the position vector of point (x,y,z), prove that $\nabla^2 r^n = n(n+1) r^{n-2}$ and hence deduce $\nabla(1/r)$.

Sol:- Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r = \sqrt{x^2+y^2+z^2} \Rightarrow r^n = (x^2+y^2+z^2)^{n/2}$$

$$\nabla r^n = \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}$$

$$= \vec{i}\left[n(x^2+y^2+z^2)^{n/2-1}(x)\right] + \vec{j}\left[n(x^2+y^2+z^2)^{n/2-1}(y)\right]$$
$$+ \vec{k}\left[n(x^2+y^2+z^2)^{n/2-1}(z)\right]$$

$$= n \cdot r^{n-2}\vec{r} \quad \text{—— (1)}$$

$$\nabla(\nabla r^n) = \left(\frac{\partial}{\partial x}\left(nx(x^2+y^2+z^2)^{\frac{n-2}{2}}\right)\right) + \frac{\partial}{\partial y}\left(ny(x^2+y^2+z^2)^{\frac{n-2}{2}}\right)$$
$$+ \frac{\partial}{\partial z}\left(nz(x^2+y^2+z^2)^{\frac{n-2}{2}}\right)$$

$$= n\left[x^2(n-2)r^{n-4} + r^{n-2} + y^2(n-2)r^{n-4} + r^{n-2} + z^2(n-2)r^{n-4} + r^{n-2}\right]$$

$$= n\left[r^2 r^{n-4}(n-2) + 3r^{n-2}\right] = n(n+1)r^{n-2}$$

$$\nabla(1/r) = \nabla(r^{-1})$$

sub $n = -1$ in (1)

$$\nabla(r^{-1}) = (-1)r^{-1-2}\vec{r} \Rightarrow \nabla(1/r) = \frac{-\vec{r}}{r^3}.$$

3) If $\vec{F} = 3xyz^2\vec{i} + 2xy^3\vec{j} - x^2yz\vec{k}$ & $f = 3x^2 - yz$ find

i) $\nabla\cdot\nabla\cdot f$ ii) div. Curl $\vec{F}$ at $(1,-1,1)$

Sol:- Given,

$$\vec{F} = 3xyz^2\vec{i} + 2xy^3\vec{j} - x^2yz\vec{k}$$

$$f = 3x^2 - yz$$

i) $\nabla f = \vec{i}\frac{\partial}{\partial x}(3x^2-yz) + \vec{j}\frac{\partial}{\partial y}(3x^2-yz) + \vec{k}\frac{\partial}{\partial z}(3x^2-yz)$

$$= \vec{i}(6x) + \vec{j}(-z) + \vec{k}(-y)$$

$$\nabla \cdot \nabla f = \frac{\partial}{\partial x}(6x) + \frac{\partial}{\partial y}(-z) + \frac{\partial}{\partial z}(-y)$$

$$= 6$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xyz^2 & 2xy^3 & -x^2yz \end{vmatrix}$$

$$= \vec{i}\,[-x^2z] - \vec{j}\,[-2xyz - 6xyz] + \vec{k}\,[2y^3 - 3xz^2]$$

$$= \vec{i}\,(-x^2z) - \vec{j}\,(-8xyz) + \vec{k}\,(2y^3 - 3xz^2)$$

$$\text{div curl } \vec{F} = \nabla \cdot \text{Curl } \vec{F}$$

$$= \frac{\partial}{\partial x}(-x^2z) + \frac{\partial}{\partial y}(8xyz) + \frac{\partial}{\partial z}(2y^2 - 3xz^2)$$

$$= -2xz + 8xz - 6xz$$

$$= -8xz + 8xz = 0.$$

TEST 3- ECE - SPECIAL FUNCTION 1

2- marks

1. Write the Bessel's equation of order zero. (K1)
2. Write the Neumann function. (K2)
3. Define recurrence relation (K2)
4. Define indicial equation on series solution when x=0 is a regular singularity
5. Write the value of $J_{1/2}$ (K1)
6. Reduce the differential equation $x^2 \frac{d^2 y}{dx^2} + a\frac{dy}{dx} + k^2 xy = 0$ to Bessel function.

16- marks

1. Solve in series $\left(1-x^2\right) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + 4y = 0$. (K5)
2. Solve in series the equation $\frac{d^2 y}{dx^2} + xy = 0$ (K3)
3. Solve in series the equation $x\frac{d^2 y}{dx^2} + \frac{dy}{dx} + xy = 0.$ (K4)

4.

Test-3 111189C103

Unit-3

2 marks

1) Write the Bessel's equation of order zero

a) The bessel's eq of order zero is:

$$n = 0; \quad x^2 \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + x^2 y = 0$$

2) Write neumann function?

a) Neumann function is also called as the

Bessel's function of second kind of order n.

It is denoted by $Y_n(x)$

$$Y_n(x) = J_n(x) \int \frac{dx}{x[J_n(x)]^2}$$

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3. Define reccurrence relation?

a) A recurrence relation is an equation which represents a sequence based on some rule. It helps in finding the subsequent term dependent upon the proceeding term.

4. Define indical equation on series solution when $x=0$ is a regular singularity?

a $x=0$ is a regular singularity point, if the normalized differential equation $y''+p(x)y'+q(x)=0$ is such that $xp(x)$ and $x^2q(x)$ are analytic at $x=0$.

$xp(x)$ and $x^2q(x)$ are analytic at $x=0$.

Then the quadratic equation obtained by equating the co-ef of lowest degree terms in x to zero of its solution is known as indical equation.

5) Write the value of $J_{1/2}$?

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,(n+r+1)}$$

put $n = 1/2$.

$$J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{\pi x}} \left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

6) Reduce the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - n^2)y = 0$ to bessel function.

a) To reduce the differential equation $x^2\frac{d^2y}{dx^2} + x\cdot\frac{dy}{dx} + (k^2x^2 - n^2)y = 0 \to ①$

to Bessel form,

put $t = kx$, so that $\frac{dy}{dx} = k\cdot\frac{dy}{dt}$ and

$$\frac{d^2y}{dx^2} = k^2\frac{d^2y}{dt^2}$$

Now ① becomes $t^2\frac{d^2y}{dx^2} + t\cdot\frac{dy}{dt} + (t^2 - n^2)y = 0$

is the required bessel's function.

16-marks

1) Solve in series $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$.

a) $x = 0$ is an ordinary point. Since, co-eff of $\frac{d^2y}{dx^2} \neq 0$ when $x = 0$.

The solution is.

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$$

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + \dots + n(n-1)a_nx^{n-2} + \dots$$

sub $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in given equation.

$$(1-x^2)\left[2a_2+6a_3x+12a_4x^2+\cdots+n(n-1)a_nx^{n-2}+\cdots\right]$$

$$-x\left[a_1+2a_2x+3a_3x^2+\cdots+na_nx^{n-1}+\cdots\right]+$$

$$4\left[a_1+a_1x+\cdots+a_nx^n\right]$$

equating co-ef of various powers of x to zero

$2a_2+4a_0=0$ {co-ef of $x^0=0$}

$a_2=-2a_0$

$6a_3-a_1+4a_1=0$ (co-ef of $x=0$)

$a_3=\frac{-a_1}{2}$

$12a_4-2a_2-2a_2+4a_2=0$ (co-ef of $x^2=0$)

$a_4=0$

$20a_5-6a_3-3a_3+4a_3=0$

$20a_5-5a_3=0$ (co-ef of $x^3=0$)

$a_5=\frac{-a_1}{8}$

$(n+2)(n+1)a_{n+2}-n(n-1)a_n-na_n+4a_n=0$

(co-ef of $x^n=0$)

$a_{n+2}=\frac{n-2}{n+1}a_n$

put $n = 4, 5, 6, 7$

$$a_6 = \frac{2}{5} a_4 = 0$$

$$a_7 = \frac{3}{6} a_5 = \frac{-a_1}{16}$$

$$a_8 = \frac{4}{7} a_6 = 0$$

$$a_9 = \frac{5}{8} a_7 = \frac{-5a_1}{128}$$

sub in y, we get required solution.

$$y = a_0 + a_1 x + (-2a_0)x^2 + \left(\frac{-a_1}{2}\right)x^3 + \left(\frac{-a_1}{8}\right)x^5 + \left(\frac{-a_1}{16}\right)x^7 + \left(\frac{-5a_1}{128}\right)x^9 + \cdots$$

$$y = a_0(1 - 2x^2) + a_1 x\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} \mp \cdots\right)$$

2) Solve in series the equation $\frac{d^2y}{dx^2} + xy = 0$.

a) $x=0$ is an ordinary point. Since co-eff of

$\frac{d^2y}{dx^2} \neq 0$ at $x = 0$; $\frac{d^2y}{dx^2} + xy = 0 \longrightarrow (1)$

The solution is

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots + n(n-1)a_n x^{n-2} + \cdots$$

sub $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in ①

$$(2a_2 + 6a_3x + 12a_4x^2 + \cdots + n(n-1)a_nx^{n-2} + \cdots)$$

$$+ x(a_0 + a_1x + \cdots + a_nx^n + \cdots) = 0$$

$$2a_2 + (6a_3 + a_0)x + (12a_4 + a_1)x^2 + (20a_5 + a_2)x^3$$

$$+ \cdots + [(n+2)(n+1)a_{n+2} + a_{n-1}]x^n + \cdots = 0$$

co-eff of various powers of x^n equal to zero

$a_2 = 0$ (co-ef of $x^0 = 0$)

$6a_3 + a_0 = 0$

$a_3 = \frac{-a_0}{6}$ (co-ef of $x = 0$)

$12a_4 + a_1 = 0$

$a_4 = \frac{-a_1}{12}$ (co-ef of $x^2 = 0$)

$20a_5 + a_2 = 0$

$a_5 = \frac{-a_2}{20} = 0$ (co-ef of $x^3 = 0$)

$(n+2)(n+1)a_{n+2} + a_{n-1} = 0$ (co-ef of $x^n = 0$)

$$a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \rightarrow (2)$$

From (2),

$$a_6 = \frac{-a_3}{6 \cdot 5} = \frac{a_0}{180}$$

$$a_7 = \frac{-a_4}{7 \cdot 6} = \frac{a_1}{504}$$

$$a_8 = \frac{-a_5}{8 \cdot 7} = 0$$

$$a_9 = \frac{-a_6}{9 \cdot 8} = \frac{-a_0}{12960}$$

sub in y, we get req solution.

$$y = a_0 + a_1 x + \left(\frac{-a_0}{6}\right) x^3 + \left(\frac{-a_1}{12}\right) x^4 + \frac{a_0}{180} x^6$$

$$+ \frac{a_1}{504} x^7 + \left(\frac{-a_0}{12960}\right) x^9 + \cdots$$

$$y = a_0\left[1 - \frac{x^3}{6} + \frac{x^6}{180} - \frac{x^9}{12960} + \right] + a_1\left[x - \frac{x^4}{12} + \frac{x^7}{504} \cdots\right]$$

3) Solve in series the eq $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$

a) S. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \rightarrow (1)$

$x = 0$ is singular point. Since, co-eff of

$\frac{d^2y}{dx^2} = 0$ at $x = 0$

The solution is;

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3}.$$

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1) a_1 x^m + \cancel{a_3 x^{m+3}} (m+2) a_2 x^{m+1}$$

$$\frac{d^2y}{dx^2} = m(m-1) a_0 x^{m-2} + m(m+1) a_1 x^{m-1} + (m+1)(m+2) a_2 x^m + \cdots$$

sub $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in ①

$$x\left[m(m-1) a_0 x^{m-2} + m(m+1) a_1 x^{m-1} + \cdots\right] + \left[m a_0 x^{m-1} + (m+1) a_1 x^m + \cdots\right] + x\left[a_0 x^m + a_1 x^{m+1} + \cdots\right] = 0$$

Lowest power of x to zero (x^{m-1})

$$a_0(m)(m-1) + m a_0 = 0.$$

$$a_0 m^2 = 0 \Rightarrow m^2 = 0$$

$m = 0, 0$ (The roots are identical)

equating co-ef of x^m, x^{m+1}, x^{m+2} to zero

$$a_1(m)(m+1) + (m+1) a_1 = 0 \quad ((\text{co-ef of } x^m)$$

$$a_1(m+1)(m+1) = 0$$

$$a_1 = 0$$

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$a_2(m+1)(m+2) + (m+2)a_2 + a_0 = 0$

$a_2\left[m+1+1\right](m+2) + a_0 = 0$ (co-eff of x^{m+1})

$$a_2 = \frac{-a_0}{(m+2)^2}.$$

$a_1 + (m+3)a_3 + (m+2)(m+3)a_3 = 0.$

(co-eff of x^{m+2})

$a_3 = 0$

$a_2 + (m+4)a_4 + (m+4)(m+3)a_4 = 0.$

$$a_4 = \frac{-a_2}{(m+4)^2} = \frac{a_0}{(m+2)^2(m+4)^2}$$ (co-eff of x^{m+3})

$$y = a_0 x^m\left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} \cdots\right] \rightarrow (2).$$

put $m=0$, the first solution is.

$$y_1 = a_0\left[1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots\right]$$

Since, the roots are identical to get second solution, partial differentiate (2) w.r.to m.

$$\frac{dy}{dm} = y\log x + a_0 x^m \left[\frac{x^2}{(m+2)^2}\left(\frac{2}{m+2}\right) - \frac{x^4}{(m+2)^2(m+4)^2}\left[\frac{2}{m+2} + \frac{2}{m+4}\right] + \cdots \right]$$

$$y_2 = \left(\frac{dy}{dm}\right)_{m=0}$$

$$= y_1 \log x + a_0 \left[\frac{x^2}{4} - \frac{3x^4}{128} + \cdots \right]$$

The complete solution is $y = c_1 y_1 + c_2 y_2$.

$$y = c_1 a_0 \left[1 - \frac{x^2}{4} + \frac{x^4}{64} \cdots \right] + c_2 y_1 \log x +$$

$$c_2 a_0 \left[\frac{x^2}{4} - \frac{3x^4}{128} + \cdots \right]$$

TEST 4- ECE- SPECIAL FUNCTION 2- SERIES SOLUTIONS

2- marks

1. Write the Fourier- Legendre expansion of f(x) from x=-1 to 1. (K1)
2. Write the polynomial $2x^2+x+3$ in terms of Legendre polynomials (K1)
3. Write the Rodrigues's formula (K1)
4. Write the generating function of Legendre polynomials (K2)
5. What is λ when $P_5(x) = \lambda(63x^5-70x^3+15x)$ a Legendre polynomial? (K2)
6. Write the Legendre's polynomial of order n (K2)

16- marks

1. Show that for any function f(x) , $\int_{-1}^{1} f(x)\, p_n(x)\, dx = \frac{1}{2^n n!}\int_{-1}^{1}\left(1+x^2\right)^n f^n(x)\, dx$. (K4)
2. Prove that $(1-x^2)\, P_n'(x) = n[P_{n-1}(x) - x\, P_n(x)]$. (K3)
3. Show that $P_n(x) = (n+1)P_{n-1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$. (K3)

4.

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Unit-4

2 marks

1) Write the fourier legendre expansion of $f(x)$ from $x=-1$ to 1?

a) If $f(x)$ be a function from $x=-1$ to 1, we can write.

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x).$$

$$\int_{-1}^{1} f(x) P_n(x) dx = c_n \int_{-1}^{1} P_n^2(x) dx = \frac{2c_n}{2n+1}$$

2) Write the polynomial $2x^2+x+8$ in terms of legendre polynomial?

a) W.K.T, $P_0(x)=1$, $P_1(x)=x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

$$\Rightarrow x^2 = \frac{2}{3} P_2(x) + \frac{1}{3}$$

$$f(x) = 2x^2+x+3$$

$$= 2\left[\frac{2}{3} P_2(x) + \frac{1}{3}\right] + P_1(x) + 3P_0(x)$$

$$= \frac{1}{3}\left[4P_2(x) + 3P_1(x) + 11P_0(x)\right]$$

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3) Write the Rodrigue's formula?

a) $$P_n(x) = \frac{1}{n!\,2^n} \cdot \frac{d^n}{dx^n}(x^2-1)^n$$

4) Write the generating function of legendre's polynomial?

a) $$(1-2xt+t^2)^2 = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

$P_n(x)$ is co-ef of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$. It is known as generating fn of legendre's polynomial.

5) What is λ when $P_5(x) = \lambda(68x^5 - 70x^3 + 15x)$ is a legendre polynomial?

a) W.K.T, $P_5(x) = \frac{1}{8}\left[63x^5 - 70x^3 + 15x\right]$.

comparing with given $P_5(x)$ we get, $\lambda = \frac{1}{8}$

6) Write legendre's polynomial of order n?

a) $$P_n(x) = \sum_{m=0}^{\mu} (-1)^m \frac{(2n-2m)!}{2^n\, m!\,(n-m)!\,(n-2m)!} \cdot x^{n-2m}$$

$\mu = \frac{n}{2}$ (or) $\frac{n-1}{2}$ which ever is an integer.

16-marks

1) Show that for any function $f(x)$

$$\int_{-1}^{1} f(x)\, P_n(x) = \frac{1}{2^n n!} \cdot \int_{-1}^{1} (1+x^2)^n f^n(x)\, dx.$$

a) Using Rodrigue's formula,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$\int_{-1}^{1} f(x)\, P_n(x)\, dx = \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n (x^2-1)^n}{dx^n} dx.$$

$$= \frac{1}{2^n n!} \left\{ \left[f(x) \frac{d^{n-1} (x^2-1)^n}{dx^{n-1}} \right]_{-1}^{1} - \int_{-1}^{1} f'(x) \frac{d^{n-1} (x^2-1)^n}{dx^{n-1}} dx. \right\}$$

$$= \frac{-1}{2^n n!} \left[\int_{-1}^{1} f'(x) \frac{d^{n-1} (x^2-1)^n}{dx^{n-1}} dx \right].$$

$$= \frac{-1}{2^n n!} \left[\left[f'(x) \frac{d^{n-2} (x^2-1)^n}{dx^{n-2}} \right]_{-1}^{1} - \int_{-1}^{1} f''(x) \frac{d^{n-2} (x^2-1)^n}{dx^{n-2}} dx \right]$$

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$$= \frac{-1}{2^n n!}\left[\int_{-1}^{1} f''(x)\, \frac{d^{n-2}(x^2-1)^n}{dx^{n-2}}\, dx\right]$$

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^n(x)(x^2-1)^n\, dx$$

(∵ Integration by parts upto (n-2) times)

$$= \frac{(-1)^{2n}}{2^n n!}\int_{-1}^{1} f^n(x)(1-x^2)^n\, dx.$$

$$= \frac{1}{2^n n!}\int_{-1}^{1} f^n(x)(1-x^2)^n\, dx.$$

2) Prove that $(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$

a) W.K.T

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x)$$

Multiply by x.

$$nxP_n(x) = x^2P_n'(x) - xP_{n-1}'(x) \rightarrow (1)$$

W.K.T,

$$P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x) \rightarrow (2)$$

Add (1) and (2)

$$P_n'(x) + nxP_n(x) = x^2P_n'(x) - \cancel{xP_{n-1}'(x)} + \cancel{xP_{n-1}'(x)} + nP_{n-1}(x)$$

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$(1-x^2) P_n'(x) = nP_{n-1}(x) - nxP_n(x)$

$$(1-x^2) P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$$

Hence proved.

3) Show that $P_n(x) = (n+1) P_{n-1}(x) = (2n+1)x P_n(x) - nP_{n-1}(x)$.

a) W.K.T,

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n \rightarrow (1)$$

To prove that, $P_n(x) = (n+1) P_{n-1}(x) = (2n+1) xP_n(x) - nP_{n-1}(x)$

Differentiate partially w.r.to 't' we get.

$$\frac{-1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum n P_n(x) t^{n-1}$$

$$(x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2) \sum n P_n(x) t^{n-1}$$

$$(x-t) \sum P_n(x) t^n = (1-2xt+t^2) \sum n P_n(x) t^{n-1}$$

equating coeff of x^n on both sides

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2nxP_n(x) + (n-1) P_{n-1}(x).$$

It follows;

$$(n+1)P_{n+1}(x) = (2n+1)x\,P_n(x) - nP_{n-1}(x)$$

$$P_n(x) = (n+1)\cdot P_{n-1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Hence proved.

TEST 5- ECE- DESIGN OF EXPERIMENT- ANOVA

2- marks

1. What are the advantages of a completely Randomised Experimental Design. (K2)
2. Write down the ANOVA table for one way classification. (K1)
3. What is the aim of design of experiments? (K2)
4. State the advantage and disadvantage of randomized block design. (K1)
5. Define ANOVA (K1)
6. What are the assumptions in analysis of variance? (K2)

16- marks

1. Perform two way ANOVA for the given below:

Plots of land	Treatment			
	A	B	C	D
I	38	40	41	39
II	45	42	49	36
III	40	38	42	42

2. The following data represent the no. of units of productions per day turned out by different workers using 4 different types of machines

	Machine			
	A	B	C	D
I	44	38	47	36
II	46	40	52	43
III	34	36	44	32
IV	43	38	46	33
V	38	42	49	39

Test whether the five men differ with respect to mean productivity and test whether the mean productivity is the same for the four different machine type.

3. The following table gives monthly sales (in thousand rupees) of a certain firm in the three states by its four salesmen. Setup the analysis of variance table and test whether there is

any significant difference (i) between sales by the firm salesmen and (ii) between sales in the three states.

States	Salesmen			
	I	II	III	IV
A	6	5	3	8
B	8	9	6	5
C	10	7	8	7

(K5)

Test-5

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Unit-5

2-marks

1) What are the advantages of completely randomised experimental design.

a) * Complete flexibility is allowed, any number of treatments and replicates may be used

* Relatively easy statistical analysis.

2) Write down the Anova table for one way classification.

a) Source of variation	Sum of squares	Degree of freedom	Mean sum of squares	Variance ratio
Between column	SSC	C-1	$MSC = \frac{SSC}{C-1}$	$f = \frac{MSC}{MSE}$
within column	SSE	N-C	$MSE = \frac{SSE}{N-C}$	$f = \frac{MSE}{MSC}$

3) What is the aim of design of experiments?

a) The aim of design of experiment is to reduce the variance in an experiment with the method of sampling.

4) State the advantages and disadvantages of randomised block design

a) <u>advantages</u>

* The precision is more in RBD
* The amount of information obtained in RBD is more as compared to CRD.

<u>disadvantages</u>

* When the number of treatments is increased; The block size will increase.
* If the block size is large maintaining homogenity is difficult.

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5) Define Anova?

a) To test equality of means, the analysis of variance technology is applied. This is one of most powerful ~~total to~~ tool to statistical analysis.

6) What are the assumptions in analysis of variance?

a) The two main assumptions are normality and homogenity.

1) Normality of Dv distribution:- The variance in each all should be approximately normally distributed. Check via skewness and kurtosis over all for each all.

2) Homogenity of Variance:- The variance in each all should be similar. Check via levers test which are generally produced as part of anova statistical output.

16-marks

1) Perform two way anova for given below

plots of land	A	B	C	D
I	38	40	41	39
II	45	42	49	36
III	40	38	42	42

Treatment.

a) Ho: There is no significant difference between treatment and plots of land

H₁: There is significant diff btw treatment and plots of land.

subtract 40 from all observations given.

	x_1	x_2	x_3	x_4	Total	x_1^2	x_2^2	x_3^2	x_4^2
y_1	-2	0	1	-1	-2	4	0	1	1
y_2	5	2	9	-4	12	25	4	81	16
y_3	0	-2	2	2	2	0	4	4	4
	3	0	12	-3	12	29	8	86	21

N - Total no. of observations = 12.

T - Total of all observations = 12.

Correction factor - $\frac{T^2}{N} - \frac{(12)^2}{12} = 12.$

$$SST = \left[\Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2\right] - \frac{T^2}{N}$$

$$= \left[29 + 8 + 86 + 21\right] - 12.$$

$$= 132.$$

$$SSC = \left[\frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1}\right] - \frac{T^2}{N}$$

$$= \left[\frac{(3)^3}{3} + \frac{(0)^2}{3} + \frac{(12)^2}{3} + \frac{(-3)^2}{3}\right] - 12.$$

$$= 42.$$

$$SSR = \left[\frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2}\right] - \frac{T^2}{N}$$

$$= \left[\frac{(-2)^2}{4} + \frac{(12)^2}{4} + \frac{(2)^2}{4}\right] - 12.$$

$$= 26.$$

$$SSE = SST - (SSC + SSR)$$

$$= 132 - [42 + 26]$$

$$= 64.$$

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Source of variance	Sum of squares	Degrees of freedom	Mean sum of squares	Variance of ratio
Between column	SSC = 42	$c-1 = 4-1$ $= 3$	$MSC = \frac{SSC}{c-1}$ $= 14$	$F_c = \frac{MSC}{MSE}$ $= \frac{14}{10.67} = 1.31$
Btw row	SSR = 26	$r-1 = 3-1$ $= 2$	$MSR = \frac{SSR}{r-1}$ $= 13$	$F_r = \frac{MSR}{MSE}$ $= \frac{13}{10.67} = 1.21$
Residual	SSE = 64	$N-C-r+1$ $= 6$	$MSE = \frac{SSE}{N-C-r+1}$ $= 10.67$	

calculated value of $F_c = 1.31$

calculated value of $F_r = 1.21$

Table value of F_c with df $(6,3)$ is 8.94

Table value of F_r with df $(6,2)$ is 19.33

Conclusion;

For F_c, C.V < T.V, H_0 is accepted at 5% level of significance. There is no significant difference between treatment.

For F_r, C.V < T.V, H_0 is accepted at 5% level of significance. There is no significant diff btw plots of land.

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2) The following data represent the no. of units of productions per day turned out by different workers using 4 diff types of machines.

	A	B	C	D
I	44	38	47	36
II	46	40	52	43
III	34	36	44	32
IV	43	38	46	33
V	38	42	49	39

Test whether the 5 men differ with respect to man productivity and test whether the mean productivity is same for four diff machine type.

a) Ho: There is no significant difference between productivity and machine.

H1: There is significant difference between productivity and machine.

Subtract 40 from all observations given.

	x_1	x_2	x_3	x_4	Total	x_1^2	x_2^2	x_3^2	x_4^2
y_1	4	-2	7	-4	5	16	4	49	16
y_2	6	0	12	3	21	36	0	144	9
y_3	-6	-4	4	-8	-14	36	16	16	64
y_4	3	-2	6	-7	0	9	4	36	49
y_5	-2	2	9	-1	8	4	4	81	1
	5	6	38	-17	20	101	28	326	139

N = Total no. of observations = 20.

T = 20.

correction Factor $= \frac{T^2}{N} = 20$.

$$SST = \left[\Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2\right] - \frac{T^2}{N}.$$

$$= [101 + 28 + 326 + 139] - 20$$

$$= 574.$$

SSC = 338.8.

SSR = 161.5.

$$SSE = SST - (SSC + SSR) = 574 - (338.8 + 161.5)$$

$$= 73.7$$

Source of variance	Sum of squares	Degrees of freedom	Mean sum of squares	variance ratio
Between column	$SSC = 338.8$	$c-1 = 4-1 = 3$	$MSC = \frac{SSC}{c-1} = 112.93$	$F_c = \frac{MSC}{MSE} = 18.39$
Between row	$SSR = 161.5$	$r-1 = 5-1 = 4$	$MSR = \frac{SSR}{r-1} = 40.37$	$F_r = \frac{MSR}{MSE} = 6.57$
Residual	$SSE = 73.7$	$N-C-r+1 = 12$	$MSE = \frac{SSE}{N-C-r+1} = 6.142$	

Table value of F_c with (12, 3) df is 8.74.

Table value of F_r with (12, 4) dif is 5.91

conclusion

For F_c, C.V > T.V, Ho is rejected at 5% level of significance. There is significant diff btw the productivity.

For F_r, C.V > T.V, Ho is rejected at 5% level of significance. There is significant diff btw the machine.

3) The following table gives monthly values of a certain form in 3 states by its 4 salesman. Test whether there is any significant diff.

i) btw sales by the firm salesman

ii) btw sales in 3 state.

states	Salesman: I	II	III	IV
A	6	5	3	8
B	8	9	6	5
C	10	7	8	7

a) Subtract 5 from all observations given.

	x_1	x_2	x_3	x_4	Total	x_1^2	x_2^2	x_3^2	x_4^2
y_1	1	0	-2	3	2	1	0	4	9
y_2	3	4	1	0	8	9	16	1	0
y_3	5	2	3	2	12	25	4	9	4
	9	6	2	5	22	35	20	14	13

H_0: There is no significant difference btw salesman and states.

H_1: There is significant difference between salesman and states.

N = Total no. of observations = 12

T = " of all " = 22

$$C.F = \frac{T^2}{N} = 40.33$$

$$SST = \left[\Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2\right] - \frac{T^2}{N}$$

$$= 41.69$$

$$SSC = \left[\frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1}\right] - \frac{T^2}{N}$$

$$= 8.33$$

$$SSR = \left[\frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2}\right] - \frac{T^2}{N}$$

$$= 12.67$$

$$SSE = SST - (SSC + SSR) = 20.67$$

Source of variance	Sum of squares	Degrees of freedom	Mean sum of squares	variance ratio
Between column	$SSC = 8.33$	$C-1 = 4-1 = 3$	$MSC = \frac{SSC}{C-1} = 2.776$	$F_C = \frac{MSC}{MSE} = 0.805$
Between row	$SSR = 12.67$	$r-1 = 3-1 = 2$	$MSR = \frac{SSR}{r-1} = 6.335$	$F_R = \frac{MSR}{MSE} = 1.834$
Residual	$SSE = 20.67$	$N-C-r+1 = 6$	$MSE = \frac{SSE}{N-C-r+1} = 3.445$	

Table value of F_c with (6,3) d.f is 8.94.

Table value of F_R with (6,2) d.f is 19.33.

Conclusion:

i) For F_c, C.V < T.V, H_0 is accepted at 5% level of significance. There is no significant diff between sales by firm salesman.

ii) For F_R, C.V < T.V, H_0 is accepted at 5% level of significance. There is no significant difference between sales in 3 states.